Dynamics of transcendental entire functions with escaping singular orbits

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Leticia Pardo Simón

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Abstract

This thesis concerns the behaviour under iteration of certain transcendental entire functions with bounded singular set that contain critical values in their escaping set. More precisely, we define *strongly postcritically separated* functions as a generalization of hyperbolic maps that in particular allows their postsingular set to be *unbounded*. In addition, the functions considered have bounded criticality in their Julia set and the postsingular points in their Julia set are "sufficiently spread". This the first time that a thorough study of transcendental entire maps with unbounded postsingular set has been conducted. We also introduce the class $\mathcal{CB}$ of transcendental entire functions with bounded singular set for which the Julia set of the disjoint type functions in their parameter space is a *Cantor bouquet*. We study the presence of *dynamic rays* in the escaping set of functions in class $\mathcal{CB}$ and conclude that they all are *criniferous*. As the main result of the thesis, for every $f \in \mathcal{CB}$ that is strongly postcritically separated, we construct a *topological model* for the action of f on its Julia set $J(f)$. In particular, we conclude that $J(f)$ is a collection of *dynamic rays* that *split* at (preimages of) critical points, together with their corresponding landing points.

Our arguments rely heavily on *expansion*. In fact, we show that strongly postcritically separated maps with bounded singular set are *expanding* in a neighbourhood of their Julia set with respect to a suitable orbifold metric. Besides using this expansion in the construction of the topological model aforementioned, it also allows us to generalize existing results for hyperbolic functions by giving criteria for the boundedness of Fatou components and local connectivity of Julia sets. As part of this study, we develop some results on *hyperbolic orbifold metrics* that might be of independent interest for future applications in holomorphic dynamics. Finally, for strongly postcritically separated functions that are additionally in the cosine family, we construct a more *explicit model* for their dynamics that we use to conclude that no two dynamic rays of the map cosh land together.

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To all of you,

¡gracias!

Contents

Chapter 1

Introduction

In one-dimensional *holomorphic dynamics* we study the behaviour under iteration of analytic functions in the complex plane. In particular, for each entire function we try to describe its set of stability, known as the *Fatou set*, as well as its complement, the *Julia set*, which is the locus of chaotic behaviour. More precisely, for an entire function f, its Fatou set $F(f)$ is defined as the set of all points $z \in \mathbb{C}$ such that $\{f^n\}_{n\in\mathbb{N}}$ form a normal family in a neighbourhood of z, and its Julia set as $J(f) := \mathbb{C} \setminus F(f)$. In this study, and more generally in *dynamical systems*, the notion of *expansion* in its various forms is fundamental. In particular, for a holomorphic function, expansion has frequently been understood in terms of a conformal metric defined on a neighbourhood of its Julia set. More specifically, a polynomial p is said to be *hyperbolic* if it is expanding with respect to a conformal metric induced on a neighbourhood of its compact Julia set $J(p)$. This is equivalent to saying that every critical value of p belongs to the basin of attraction of a periodic cycle, and in particular its orbit lies in the Fatou set $F(p)$ [DH84, Theorem 1, page 21]. As a consequence of this expansion, whenever the Julia set of a hyperbolic polynomial is connected, then it is also locally connected [DH84, Proposition 4, page 19].

For transcendental entire maps, infinity is an essential singularity and thus their Julia sets are no longer compact. Still, with slight modifications on the notion of expansion, that in particular requires the hyperbolic metric to be defined in a punctured neighbourhood of infinity, a definition and characterization of *hyperbolic transcendental maps* are analogous to those in the polynomial case. See [RGS16, Theorem and Definition 1.3] and Definition 3.1. Again, expansion of hyperbolic transcendental maps was used in [BFRG15] to draw conclusions on the topology of their Julia and Fatou sets.

Regarding expansion arguments, it is crucial for a hyperbolic map f that both its set of singular values $S(f)$, that is, of singularities of the inverse f^{-1}, and the closure of its forward orbit, called the *postsingular set* $P(f) := \overline{\bigcup_{n\geq 0} f^n(S(f))}$, are contained in its Fatou set. This is because then, all iterates of f act as a covering map in a neighbourhood of $J(f)$ that does not intersect $P(f)$, a neighbourhood where the hyperbolic metric sits. Even if such a neighbourhood no longer exists for *subhyperbolic polynomials*, that is, those for which $P(p) \cap J(p)$ is finite, still Douady and Hubbard were able to extend these ideas to this more general setting. More precisely, inspired by work of Thurston [Thu84a], they overcame the presence of postsingular points in the Julia set of subhyperbolic polynomials by considering $J(p)$ as a subset of a Riemann orbifold on which p acts as an orbifold covering map. In particular, they proved subhyperbolic polynomials to be expanding with respect to a corresponding orbifold metric [DH84], see §2.2 for basic definitions on orbifold metrics. Thanks to this expansion, they showed that the aforementioned result on local connectivity of Julia sets for hyperbolic polynomials generalizes to subhyperbolic ones [DH84, Proposition 4, page 19].

The notion of subhyperbolicity for transcendental maps was first introduced by Mihaljević-Brandt in [MB12]. A transcendental entire map f is said to be *subhyperbolic* if $P(f) \cap F(f)$ is compact and $P(f) \cap J(f)$ is finite. For a transcendental entire function, the presence of asymptotic values or critical points with arbitrarily large local degree in its Julia set prevents its Julia set to be successfully considered a subset of an orbifold [MB12, Proposition 3.6]. However, orbifold expansion is achieved in [MB12, Theorem 4.1] for subhyperbolic functions for which this does not occur. That is, subhyperbolic maps with *bounded criticality* on their Julia sets, which are called *strongly subhyperbolic*.

Note that since the postsingular set of subhyperbolic transcendental maps is bounded, all these maps belong to the broadly studied *Eremenko-Lyubich class* $\mathcal{B}$. This class consists of all transcendental entire functions with bounded singular set, and this resemblance to the polynomial case has made this class a target of study in transcendental dynamics. Moreover, the fact that for subhyperbolic maps the postsingular set is also bounded is decisive in the arguments concerning estimates on orbifold metrics in [DH84, MB12].

In this thesis we generalize strongly subhyperbolic functions to a class of functions that contain critical values escaping to infinity, and thus their postsingular set might be **unbounded**. More precisely, we say that a transcendental entire function f is *postcritically separated* if $P(f) \cap F(f)$ is compact and $P_J := J(f) \cap P(f)$ is **discrete**. If in addition f has bounded criticality in $J(f)$, there is a uniform bound on the number of critical points in the orbit of any $z \in J(f)$, and there is $\epsilon > 0$ so that for any distinct $z, w \in P_J$, $|z - w| \geq \epsilon \max\{|z|, |w|\}$, we say that f is *strongly postcritically separated.* See Definition 3.3. We note that these maps might have unbounded singular set, but expansion is achieved for those that are additionally in class $\mathcal{B}$. For each strongly postcritically separated map $f \in \mathcal{B}$ in this class, we associate a pair of orbifolds $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ whose *underlying surfaces* are respective neighbourhoods of $J(f)$, and so that we can *extend* f to be an orbifold covering map between them. With slight abuse of notation, we also denote this map between orbifolds by f, see §2.2 for more details. Our first result is the following:

Theorem 1.1 (Orbifold expansion for strongly postcritically separated maps)**.** *Let $f \in \mathcal{B}$ be a strongly postcritically separated map. Then, there exist a constant $\Lambda > 1$ and a pair of hyperbolic orbifolds $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ such that $f\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map,*

$$\|Df(z)\|_{\mathcal{O}} := |f'(z)| \cdot \frac{\rho_{\mathcal{O}}(f(z))}{\rho_{\mathcal{O}}(z)} \geq \Lambda \tag{1.0.1}$$

whenever the quotient is defined, and $J(f)$ is contained in the underlying surfaces of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.

We use Theorem 1.1 to generalize some of the results in [BFRG15] on the topology of Julia and Fatou sets of hyperbolic functions to the larger class of strongly postcritically separated maps. The first one is a generalization of [BFRG15, Theorem 1.2]:

Theorem 1.2 (Bounded Fatou components)**.** *Let $f \in \mathcal{B}$ be strongly postcritically separated. Then the following are equivalent:*

(a) every component of $F(f)$ is a bounded Jordan domain;

(b) f has no asymptotic values and every component of $F(f)$ contains at most finitely many critical points.

As a consequence of this theorem, we obtain the following result on local connectivity of Julia sets, that generalizes [BFRG15, Corollary 1.8].

Corollary 1.3 (Bounded degree implies local connectivity)**.** *Let $f \in \mathcal{B}$ be strongly postcritically separated with no asymptotic values. Suppose that there is a uniform bound on the number of critical points, counting multiplicity, in the Fatou components of f. Then $J(f)$ is locally connected.*

The next result provides further sufficient conditions for the local connectivity of Julia sets. Compare to [BFRG15, Corollary 1.9(a)].

Corollary 1.4 (Locally connected Julia sets)**.** *Let $f \in \mathcal{B}$ be strongly postcritically separated with no asymptotic values, suppose that every component of $F(f)$ contains at most one critical value, and that the multiplicity of the critical points of f is uniformly bounded. Then $J(f)$ is locally connected.*

Another strategy frequently used to understand the dynamics of a map is to relate them to those of a "simpler" or "already understood" map. For example, if p a polynomial of degree $d \geq 2$, then Böttcher's Theorem provides a conjugacy between p and the simpler map $z \mapsto z^d$ (whose Julia set is the unit circle $\partial\mathbb{D}$) in a neighbourhood of infinity. Whenever all the orbits of the critical points of p are bounded (or equivalently $J(p)$ is connected), this conjugacy can be extended to a biholomorphic map between $\mathbb{C} \setminus \overline{\mathbb{D}}$ and the basin of infinity of p. This conjugacy allows to define *dynamic rays* for p as the curves that arise as preimages of rays from $\partial\mathbb{D}$ to ∞ under such a conjugacy, and provide a natural foliation of the points of p that escape to infinity. In many important cases, the Julia set of a polynomial p is locally connected and each ray has a unique accumulation point in $J(p)$. Then we say that each ray *lands*, and this limiting behaviour of the dynamic rays has been used with great success to provide a combinatorial description of the dynamics of p in $J(p)$. For example, in this situation, Douady [Dou93] constructed a *topological model* for $J(p)$ as a "pinched disc", that is, as the quotient of $\partial\mathbb{D}$ by a natural equivalence relation.

Since for transcendental entire functions infinity is an essential singularity, Böttcher's Theorem no longer applies. Still, it is known that for functions in class $\mathcal{B}$ and of finite order of growth, every point that escapes to infinity can be connected to infinity by an escaping curve, subsequently called *dynamic ray* by analogy with the polynomial case [Bar07, RRRS11]. We say that a function f is of *finite order* if $\log\log|f(z)|$ is of the order of $\log|z|$ when $|z| \to \infty$ (e.g. $f(z) := \exp(z^d)$ has order d), and the *escaping set* $I(f)$ of a transcendental map f is the set of points that escape to infinity under iteration. Given the importance in our results, we now fix the definition of dynamic ray for transcendental entire functions that we have adopted:

Definition 1.5 (Dynamic rays for transcendental maps [RRRS11, Definition 2.2])**.** Let f be a transcendental entire function. A *ray tail* of f is an injective curve $\gamma : [t_0, \infty) \to I(f)$, with $t_0 > 0$, such that

- for each $n \geq 1$, $t \mapsto f^n(\gamma(t))$ is injective with $\lim_{t\to\infty} f^n(\gamma(t)) = \infty$;
- $f^n(\gamma(t)) \to \infty$ uniformly in t as $n \to \infty$.

A *dynamic ray* of f is a maximal injective curve $\gamma : (0, \infty) \to I(f)$ such that the restriction $\gamma_{|[t,\infty)}$ is a ray tail for all $t > 0$. We say that γ *lands* at z if $\lim_{t\to 0^+} \gamma(t) = z$, and we call z the *endpoint* of γ.

As we have remarked before, for a transcendental function, both its singular and postsingular set play a very important role in its dynamics. In fact, all previously known results for which a complete topological description of the Julia set in terms of *existence* and *landing* of rays is achieved, are for functions whose postsingular set is bounded (e.g. [Sch07, BJR12, Rem09, MB12]). Note that for polynomials, the orbit of any point, and in particular of any critical value, is either bounded or converges to the superattracting fixed point at infinity. Even in the case when some singular values escape to infinity, it is still possible to define dynamic rays as the orthogonal trajectories of level curves for the Green's function, and rays can be extended when they hit critical points using Green's function in a somehow natural way. See for example [GM93, Appendix A] or [Kiw97, Section 2.2].

However, for transcendental entire maps, orbits necessarily interact differently with the essential singularity at infinity, leading to the trichotomy of bounded orbits, escaping orbits and those neither bounded nor escaping (*bungee set* [OS16]). Hence, a priori it is not obvious what to expect concerning dynamic rays when for a transcendental map f, $P(f)$ is unbounded, not even in the case when $P(f) \subset I(f)$. Moreover, unboundedness of $P(f)$ leads to the additional challenge that $I(f)$ might contain critical values, and thus some dynamic rays would *split* at critical points, potentially compromising the landing of rays.

This phenomenon can be illustrated with the function $f = \cosh$. In this case, $S(f) = \{-1, 1\}$ and $P(f)$ equals $S(f)$ together with the orbit of $f(-1) = f(1)$, that consists of a sequence of positive real points converging to infinity at an exponential rate. The existence of dynamic rays for f is known. Note that 0 is a critical point and it is easy to check that $(-\infty, 0]$ and $[0, \infty)$ are both ray tails. The vertical segments $[0, -i\pi/2]$ and $[0, i\pi/2]$ are mapped univalently to $[0, 1] \subset \mathbb{R}$, and thus, the union of each of them with each of the ray tails $(-\infty, 0]$ and $[0, \infty)$ forms a different ray tail. In particular, we can think of such structure as four ray tails that partially overlap pairwise. See Figure 15. Their endpoints $-i\pi/2$ and $i\pi/2$ are preimages of 0, and so the structure described has a preimage around each of them, leading again to two possible extensions of each ray tail.

We are able to show for a class of functions that includes cosh that all their ray tails can be extended in a systematic and converging way that results in the following theorem:

Theorem 1.6 (Landing of rays for functions with escaping singular orbits)**.** *Let $f \in \mathcal{B}$ be a finite composition of functions of finite order. Suppose that $S(f) \cap J(f)$ is a finite collection of points that escape to infinity so that $|w-z| \geq \epsilon \max\{|z|, |w|\}$ for some $\epsilon > 0$ and all $z, w \in P(f) \cap J(f)$. Then, every dynamic ray of f lands and every point in $J(f)$ is either on a dynamic ray or it is the landing point of one of such rays.*

A more ambitious problem to solve for transcendental entire functions is to find a topological model for the structure (and even the dynamics) on their Julia set, as an analog of to Douady's "Pinched Disc Model" for polynomials. In particular, the existence of such a model leads to a better understanding of their dynamics. This has been achieved for certain functions in class $\mathcal{B}$. The seminal work in this direction is [AO93], where it is shown that the Julia set of certain exponential and sine maps is homeomorphic to a topological object called a *straight brush* (see Definition 4.25). When this occurs, the Julia set is said to be a *Cantor bouquet*. (Compare with [BDD$^+$01, Rem06] for other parameters in the exponential family). Subsequently, it was shown in [BJR12] that if f is of *disjoint type* (i.e., $P(f) \Subset F(f)$ and $F(f)$ is connected) and of finite order, then $J(f)$ is a Cantor bouquet.

If $f \in \mathcal{B}$, then for $\lambda \in \mathbb{C}$ with $|\lambda|$ small enough, the function λf is of disjoint type. In particular, λf is in the *parameter space* of f, that is, f and λf are quasiconformally equivalent. Thus, their dynamics near infinity are related by a certain analogue of Böttcher's Theorem for transcendental maps [Rem09]. One might regard disjoint type functions as the simplest type of functions that lie in the parameter space of f, and thus playing an analogous role for f as $z \mapsto z^d$ does for a polynomial of degree d. This idea was used in [Rem09] (resp. [MB12]), where a semiconjugacy between a hyperbolic (resp. strongly subhyperbolic) map and a disjoint type map on their parameter space is built. With the additional assumption that the functions are of finite order, using that then the Julia set of the disjoint type function is a Cantor bouquet by the aforementioned result in [BJR12], they provide a topological description of Julia sets as *Pinched Cantor bouquets*, and thus a collection of landing rays. See also [ARGS19] for a generalization of this result for *strongly geometrically finite* maps.

In this thesis we obtain analogous results for certain strongly postcritically separated functions. In particular and more generally than in Theorem 1.6, these functions might contain postsingular points in their Julia set that are not escaping (Observation 3.2). In order to provide a topological model for our functions, rather than assuming that f is of finite order and using that a disjoint type map has a Cantor bouquet Julia set, we consider the more general class of functions

$$\mathcal{CB} := \left\{ \begin{array}{c} f \in \mathcal{B}: \text{ exists } \lambda \in \mathbb{C} : g_\lambda := \lambda f \text{ is of disjoint type} \\ \text{and } J(g_\lambda) \text{ is a Cantor bouquet} \end{array} \right\}.$$

In particular, we investigate in §4.3 some interesting dynamical properties of maps in this class. For example, using results from [Rem09], we show the existence of dynamic rays in their Julia sets. A transcendental entire map f is *criniferous* if every $z \in I(f)$ is eventually mapped to a ray tail. See Definition 2.27.

Theorem 1.7. *The class $\mathcal{CB}$ is closed under iteration and all maps in $\mathcal{CB}$ are criniferous.*

The main result in this thesis concerns strongly postcritically separated maps in $\mathcal{CB}$: we construct a topological model for the action of each such function f on its Julia set. Since $S(f) \cap I(f)$ might not be empty, as occurs for the map cosh, our model must reflect the splitting of rays at critical points, and so the topological structure of $J(f)$ can no longer be a (Pinched) Cantor bouquet. For that reason, for each $f \in \mathcal{CB}$, we choose any $g := \lambda f$ of disjoint type for $\lambda \in \mathbb{C}$ with $|\lambda| \ll 1$, and define our model space as two copies of $J(g)$, that is,

$$J(g)_\pm := J(g) \times \{-, +\},$$

with a *special topology* that preserves the order of rays at infinity, see §4.5. We let $I(g)_\pm := I(g) \times \{-, +\}$, and define the model map $\tilde{g}\colon J(g)_\pm \to J(g)_\pm$ to act as g on the first coordinate and as the identity on the second. We show the following:

Theorem 1.8 (Semiconjugacy to model space)**.** *Let $f \in \mathcal{CB}$ and strongly postcritically separated. Then, there exists a continuous surjective function*

$$\varphi : J(g)_\pm \to J(f) \tag{1.0.2}$$

so that $f \circ \varphi = \varphi \circ \tilde{g}$. Moreover, $\varphi(I(g)_\pm) = I(f)$.

A more detailed version of Theorem 1.8 is Theorem 4.64, where more information on the properties of the map φ is given. Since all functions considered in Theorem 1.6 are in class $\mathcal{CB}$ and strongly postcritically separated, this theorem now follows from the following corollary.

Corollary 1.9 (Landing of rays for strongly postcritically separated functions in $\mathcal{CB}$)**.** *Under the assumptions of Theorem 1.8, every dynamic ray of f lands and every point in $J(f)$ is either on a dynamic ray or it is the landing point of at least one such ray.*

In Chapter 5 we focus on the study of the cosine family, that is, maps of the form $z \mapsto ae^z + be^{-z}$ for $a, b \in \mathbb{C}^*$. More specifically, we define an *explicit model* $(J(\mathcal{F}), \mathcal{F})$ for their dynamics, so that the first coordinate of $J(\mathcal{F})$ codes the exponential growth of the functions away from the imaginary axis, and the second coordinate its orbit with respect to a partition of the plane into almost-horizontal strips of 2π height, see Definition 5.13. In Theorem 5.21 we conjugate this model to any disjoint type cosine map. Then, we consider two copies of it and construct a new model $(J(\mathcal{F})_{\pm}, \widetilde{\mathcal{F}})$. As an application of Theorem 4.64 we obtain the following result:

Theorem 1.10 (Explicit model for strongly postcritically separated cosine maps)**.** *Let f be in the cosine family and strongly postcritically separated. Then, there exists a continuous surjective function $\hat{\varphi}\colon J(\mathcal{F})_{\pm} \to J(f)$ so that $f \circ \hat{\varphi} = \hat{\varphi} \circ \widetilde{\mathcal{F}}$.*

See Theorem 5.27 for a more detailed version of Theorem 1.10. Then, we focus on the dynamics of the strongly postcritically separated maps cosh and $\cosh^2$, providing an explicit combinatorial description of the equivalence classes of points that have the same image under $\hat{\varphi}$, and concluding that no two of their dynamic rays land together, see Proposition 5.34.

In the course of proving the major results of this thesis, we have further obtained others that we believe to be of interest on their own and for future applications: firstly, as a step towards constructing the map φ in Theorem 1.7, we study properties of functions whose Julia set is a Cantor bouquet. In a rough sense, a Cantor bouquet consists of an uncountable collection of disjoint curves, called *hairs*, tending to infinity and satisfying certain density condition (see Definition 4.25). We have obtained the following result on projections of Julia sets that are Cantor bouquets to subsets of points whose orbit remains in a neighbourhood of infinity.

Theorem 1.11 (Continuous projection for Cantor bouquets)**.** *Let g be a disjoint type function whose Julia set is a Cantor bouquet, and for each $R > 0$ define*

$$J_R(g) := \{z \in J(g) : |g^n(z)| \geq R \text{ for all } n \geq 1\}.$$

Then, there exists a continuous function $\pi\colon J(g) \to J_R(g)$ such that for each hair η of the Cantor bouquet $J(g)$, there exists a point $z_\eta \in \eta$ such that π acts as the *identity map for all points in η with greater potential than that of z_η, and the image of the rest of the points in η equals z_η.*

See 4.29 and Theorem 4.31 for the definition of the function π and a more precise version of Theorem 1.11.

In order to successfully associate orbifolds to a holomorphic function so that some analogue of Theorem 1.1 holds, the set of ramified points of the orbifolds, and hence the set of singularities of the orbifold metrics, must contain the postsingular points of the function that are also in its Julia set. See the discussion at the beginning of §3.3. Since for strongly postcritically separated functions, these points might tend to the essential singularity at infinity, we require global estimates of the densities of metrics on hyperbolic orbifolds, in particular generalizing some known estimates for metrics on hyperbolic domains. These estimates, that hold for orbifolds $\widetilde{\mathcal{O}}$, $\mathcal{O}$ for which the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, come in terms of the *boundary of* $\widetilde{\mathcal{O}}$ *in* $\mathcal{O}$, denoted $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$. That is, the set of boundary points of the underlying surface of $\widetilde{\mathcal{O}}$ that are in $\mathcal{O}$ together with those points that have greater ramification value in $\widetilde{\mathcal{O}}$ than in $\mathcal{O}$. See Definition 3.6.

Theorem 1.12 (Estimates on relative densities)**.** *Let $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ be hyperbolic orbifolds such that the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, and denote by $\rho_{\widetilde{\mathcal{O}}}$ and $\rho_{\mathcal{O}}$ their respective densities. If R is the $\mathcal{O}$-distance between $z \in \widetilde{\mathcal{O}}$ and $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$, then*

$$1 < \frac{e^R}{\sqrt{e^{2R}-1}} \leq \frac{\rho_{\widetilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} \leq 1 + \frac{2}{e^R - 1}, \tag{1.0.3}$$

whenever the quotient is defined.

We also show in this thesis that whenever singularities of an orbifold metric are "continuously perturbed", the orbifold metric of the new orbifold is a "continuous perturbation" of the metric of the original orbifold. See Theorem 3.9. In particular, this result has the following implication.

Theorem 1.13 (Distances are uniformly bounded across certain orbifolds)**.** *Given a compact subset A of a Jordan domain U and constants $\epsilon > 0$ and $c, M \in \mathbb{N}_{\geq 1}$, there exists a constant $R := R(U, A, \epsilon, c, M) > 0$ such that for every orbifold $\mathcal{O}$ with underlying surface U and at most M ramified points, each with ramification value smaller than or equal to c, and such that the Euclidean distance between any two of them is at least ϵ, it holds that*

$$d_{\mathcal{O}}(p, q) < R \quad \textit{for every} \quad p, q \in A.$$

1.1 Structure of the thesis

Chapter 2 collects the background and introductory results that we shall use throughout the thesis. More specifically, we start by setting some basic notation (§2.1) and providing background on Riemann orbifolds (§2.2) and holomorphic dynamics (§2.3). Then, we focus on some broadly used techniques when studying transcendental entire functions in class $\mathcal{B}$: in §2.4 we define *(external) addresses* in terms of *fundamental domains*, a combinatorial tool that allows to code the orbit of points that remain in a neighbourhood of infinity in a string of symbols, and thus providing symbolic dynamics. We provide the set of addresses with a topology and study for criniferous functions their sets of points sharing the same external address. In particular, we show in Theorem 2.30 that those sets are ray tails or dynamic rays with their endpoints. Then, in §2.5 we recall the *logarithmic change of variables* for functions in class $\mathcal{B}$ and prove some basic properties we will use later on.

Chapter 3 regards the expansion result for strongly postcritically separated maps and its immediate consequences. In §3.1 we provide the formal definition of this class of maps, their basic properties and give some examples. §3.2 studies orbifold metrics and includes the proofs of Theorems 1.12 and 1.13. Using these results, in §3.3 we construct for each strongly postcritically separated map a pair of dynamically associated orbifolds that we use to prove Theorem 1.8. §3.4 contains the proofs of the results on Fatou components and local connectivity of Julia sets, that is, Theorem 1.2 and Corollaries 1.3 and 1.4. These proofs will easily follow from the study of *periodic* Fatou components: Theorem 3.17 gives several conditions equivalent to the boundedness of a periodic Fatou component. Finally, we introduce in §3.5 a modified notion of *homotopy* for which we obtain an analogue of the Homotopy Lifting property for certain class of curves that contain postsingular points of entire functions, Proposition 3.24. Moreover, we show in Corollary 3.27 that if U is a bounded set of a hyperbolic orbifold such that $P(f) \subset U$ is finite and there is a dynamic ray of f landing at each of those postsingular points, then there exists a constant μ such that for any piece of dynamic ray of f contained in U, we can find a curve on its "modified homotopy class" with orbifold length at most μ; a result of great value for expansion arguments.

The goal of Chapter 4 is to prove Theorem 1.8. In §4.1 we assign symbolic dynamics to certain criniferous functions by generalizing the concept of external address from §2.4: we define *signed (external) addresses* and show that for many criniferous functions, all points in their escaping set have at least two signed addresses. Signed addresses are defined in terms of some curves of points sharing that address called *canonical tails*. Next, in §4.2 we introduce the concept of *fundamental hands* as preimages of certain subsets of fundamental domains on which inverse branches are well-defined, and so that for each canonical tail, we can find an inverse branch that contains the tail on its image. In §4.3 we study general properties of functions whose Julia set is a Cantor bouquet and prove Theorem 1.11. It is in §4.4 where we study the main properties of functions in class $\mathcal{CB}$. In particular, we prove Theorem 1.7 using results from [Rem09] that provide a semiconjugacy near infinity between $f \in \mathcal{CB}$ and any disjoint type g belonging to its parameter space. We continue by defining and studying in §4.5 the *topological model* for functions in $\mathcal{CB}$ that concerns Theorem 1.8. In §4.6 we combine all results and tools developed in the previous sections to obtain the desired *semiconjugacy* from Theorem 1.8 as a limit of successively better approximations, and hence we prove Theorem 1.8 and Corollary 1.9.

Finally, Chapter 5 includes several applications of the main results in Chapters 3 and 4: in §5.1 we study the dynamics of *cosine maps* and construct models for their dynamics in different subsets of their Julia sets. In particular, we provide the proof of a more detailed version of Theorem 1.10, as well as a combinatorial description of the overlapping of dynamic rays of the maps cosh and $\cosh^2$, concluding in Proposition 5.34 that no two of their dynamic rays land together. We conclude in §5.2 with some remarks and open questions that arise from the work on this thesis.

Chapter 2

Preliminaries

The notation fixed in §2.1 will be used frequently throughout the whole thesis; any other specific notation will be fixed when relevant. §2.2 includes the basic definitions and background on Riemann orbifolds needed to understand the results in Chapter 3. §2.3 collects some basic results on holomorphic dynamics, and hence the reader familiar with the field might skip it. §2.4 and §2.5 are required to follow Chapters 4 and 5. While §2.4 fixes notation and results crucial for the understanding of Chapter 4, and more specifically §4.1, §2.5 is only used for references (Theorem 4.44) and in the proof of Theorem 4.31, so again might be disregarded.

2.1 Basic notation

As introduced in Chapter 1, the Fatou, Julia and escaping sets of an entire function f are denoted by $F(f)$, $J(f)$ and $I(f)$ respectively. Its set of critical values is denoted by $\mathrm{CV}(f)$ and that of its asymptotic values by $\mathrm{AV}(f)$. $\mathrm{Crit}(f)$ is its set of critical points (see Definition 2.12). The set of singular values of f is $S(f)$ and $P(f)$ is its postsingular set. Moreover, $P_J := P(f) \cap J(f)$ and $P_F := P(f) \cap F(f)$, and in §4.2 we use $S_I := S(f) \cap I(f)$. For each $n \geq 0$, f^n denotes the n-th iterate of f, that is, $f^n := f \circ \overset{n}{\cdots} \circ f$. We denote the complex plane by $\mathbb{C}$, the Riemann sphere by $\widehat{\mathbb{C}}$ and the upper half-plane by $\mathbb{H}$. A disc of radius ϵ centred at a point p will be $\mathbb{D}_\epsilon(p)$ or $B_\epsilon(p)$, the unit disc centred at 0 is abbreviated as $\mathbb{D}$, and $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. We will indicate the closure of a domain either by $\overline{U}$ or $\mathrm{cl}(U)$ in such a way that it will be clear from the context, and unless otherwise stated, these closures must be understood to be taken in $\mathbb{C}$. For sets, $A \Subset B$ means that A is compactly contained in B. The annulus with radii $a < b$ and the vertical strip between $x = a$ and $x = b$ will be denoted by

$$A(a, b) := \{w \in \mathbb{C} : a < |w| < b\} \quad \text{and} \quad V(a, b) := \{w \in \mathbb{C} : a < \mathrm{Re}(w) < b\}$$

respectively. For a holomorphic function f and a set A, $\operatorname{Orb}^-(A)$ and $\operatorname{Orb}^+(A)$ are the backward and forward orbit of A under f. That is,

$$\operatorname{Orb}^-(A) := \bigcup_{n=0}^{\infty} f^{-n}(A) \quad \text{and} \quad \operatorname{Orb}^+(A) := \bigcup_{n=0}^{\infty} f^{n}(A).$$

Given a sequence $\{n_i\}_i$ of natural numbers, we write $\operatorname{lcm}\{n_i\}$ for their least common multiple. The cardinality of a set A will be denoted by $\#A$, and the cardinality of a union of sets A and B by $\#(A \cup B)$. For a function $f\colon A \to B$, its restriction to a subset $C \subset A$ is denoted by $f|_C\colon C \to B$. We conclude any proof of a claim with the symbol $\triangle$, and the rest of proofs with the symbol $\square$.

For an arc $\gamma \subset \mathbb{C}$, that in particular might be a dynamic ray, we say that a point $z \in \gamma$ has *potential* t if $\gamma(t) = z$ for a given parametrization of γ. If γ is unbounded and its parametrization is not specified, we shall assume it is parametrized from $[0, \infty)$ or $(0, \infty)$, depending on whether γ includes a finite endpoint or not. This induces a total order in the points of γ: we can say for any $z, w \in \gamma$ that $z < w$ if the potential of z is smaller than the potential of w. Thus, we sometimes use expressions like "point of greatest potential" to refer to points in γ. Moreover, the concatenation of two arcs γ and β at a common endpoint will be denoted by $\gamma \cdot \beta$. In particular, curves might be degenerate.

Numbering. Sections are labelled with two numbers "$X.Y$": X indicates the chapter and Y the section within the chapter. Seeking ease of exposition, we have opted for not numbering any subsection. All statements, i.e. theorems, propositions, definitions and observations, are labelled as $X.Z$, with X for the chapter and Z providing a number for the statement (and thus there will not be a reference for the section they belong to). In addition, we have numbered some discussion paragraphs for ease on reference, which will follow the labelling of statements. Since we use two sequences both for statements and sections, we will make clear to the reader if we are referring to a section by always using the symbol "§" before the numbering in a reference. In contrast and in order to differentiate equations from statements, we use three numbers in the label of equations: $X.Y.Z$, where X indicates the chapter, Y the section and Z is the sequence of numbers that distinguish equations within sections. Figures will be labelled only with one number.

2.2 Background on Riemann orbifolds

An *orbifold* is a space that is locally represented as a quotient of an open subset S of $\mathbb{R}^n$ by a linear action of a finite group (see [Thu84a, Chapter 13]). For the purposes of this thesis, we are only interested in orbifolds modelled over Riemann surfaces. In this case, orbifolds are conveniently totally characterized by the surface S together with a map that "marks" a discrete set of points of S. For a more detailed introduction to this particular case we refer the interested reader to [McM94, Appendix A] and [Mil06, Chapter 19 and Appendix E]. For the case when the orbifold is constructed over a 2-sphere see also [BM17, Appendix A.9].

Definition 2.1 (Riemann orbifold)**.** A *Riemann orbifold* is a pair (S, ν) consisting of a Riemann surface S, called the *underlying surface*, and a *ramification map*[1] $\nu\colon S \to \mathbb{N}_{\geq 1}$ such that the set

$$\{z \in S \ : \ \nu(z) > 1\}$$

is discrete. A point $z \in S$ for which $\nu(z) > 1$ is called a *ramified* or *marked point*, and $\nu(z)$ is its *ramification value*. If $\nu(z) = 1$ we say that z is *unramified*. The *signature* of an orbifold is the list of values that the ramification map ν assumes at the ramified points, where each of them is repeated as often as it is assumed by ν.

Remark. We shall often use the term "orbifold" synonymously with "Riemann orbifold". Note that a traditional Riemann surface is a Riemann orbifold with ramification map $\nu \equiv 1$. In some cases, we will allow underlying surfaces to be disconnected, and hence certain properties should be understood component-wise.

In order to define holomorphic maps between orbifolds, we recall the following definitions: a map $f : \tilde{S} \to S$ between Riemann surfaces is a *branched covering map* if every $z \in S$ has a connected neighbourhood $U \ni z$ such that f maps any component of $f^{-1}(U)$ onto U as a proper map. Recall that a f is *proper* if the preimage $f^{-1}(K)$ of any compact set $K \subset U$ is a compact subset of $\tilde{S}$.

Definition 2.2 (Holomorphic and covering orbifold maps)**.** Let $\widetilde{\mathcal{O}} = (\widetilde{S}, \tilde{\nu})$ and $\mathcal{O} = (S, \nu)$ be Riemann orbifolds. A *holomorphic map* $f\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is a holomorphic map $f\colon \widetilde{S} \to S$ between the underlying Riemann surfaces such that

$$\nu(f(z)) \text{ divides } \deg(f, z) \cdot \tilde{\nu}(z) \quad \text{ for all } \quad z \in \widetilde{S}. \tag{2.2.1}$$

[1] Unlike in other texts, we only allow the ramification map to take finite values.

If in addition $f\colon \widetilde{S} \to S$ is a branched covering map such that

$$\nu(f(z)) = \deg(f, z) \cdot \tilde{\nu}(z) \quad \text{for all} \quad z \in \widetilde{S}, \tag{2.2.2}$$

then $f\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is an *orbifold covering map*. If there exists an orbifold covering map between $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ and in addition $\widetilde{S}$ is simply-connected, then $\widetilde{\mathcal{O}}$ a *universal covering orbifold* of $\mathcal{O}$ and f is a *universal covering map*.

We note that an orbifold covering map needs not be a between the underlying surfaces. Indeed, that will be the most frequent case for us.

Observation 2.3 (Lifts of covering maps [Mil06, Lemma E.2])**.** Let $\widetilde{\mathcal{O}}, \mathcal{O}$ be a pair of orbifolds with universal covering orbifolds. Then $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map if and only if it lifts to a conformal isomorphism between the universal covering orbifolds.

Remark. With slight abuse of notation, we will sometimes write $z \in \mathcal{O}$ to indicate that z belongs to the underlying surface of $\mathcal{O}$. Similarly, given a holomorphic map f between orbifolds, we also denote by f the holomorphic map between their underlying surfaces.

As a generalization of the Uniformization theorem for Riemann surfaces, with only two exceptions, every Riemann orbifold has a universal covering orbifold:

Theorem 2.4 (Uniformization of Riemann orbifolds)**.** *Let $\mathcal{O} = (S, \nu)$ be a Riemann orbifold for which S is connected. Then $\mathcal{O}$ has no universal covering orbifold if and only if $\mathcal{O}$ is isomorphic to $\widehat{\mathbb{C}}$ with signature (l) or (l, k), where $l \neq k$. In all other cases the universal cover is unique up to a conformal isomorphism over the surface S and given by either $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. In particular, if $S \subsetneq \mathbb{C}$ and $\#(\widehat{\mathbb{C}} \backslash S) > 2$, then $\mathcal{O}$ is covered by $\mathbb{D}$.*

In analogy to Riemann surfaces, we call an orbifold $\mathcal{O}$ *elliptic, parabolic* or *hyperbolic* if all of its connected components are covered by $\widehat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{D}$ respectively. A more detailed version of this theorem can be found in [McM94, Theorem A2]. For the proof in the more general case see [Thu84a, Proposition 13.2.4].

2.5 (Orbifold metric)**.** Theorem 2.4 allows us to induce a metric on those orbifolds that have a universal cover as the pushforward of the spherical, Euclidean or hyperbolic metric of their universal cover. More precisely, let $\mathcal{O} = (S, \nu)$ be an orbifold that has universal covering surface $C \in \{\mathbb{C}, \hat{\mathbb{C}}, \mathbb{D}\}$, and let $\rho_C(z)|dz|$ be a complete conformal metric on C. By pushing forward this metric by an orbifold covering map, we obtain a Riemannian metric on $\mathcal{O}$, that we denote by $\rho_{\mathcal{O}}(w)|dw|$ and call the *orbifold metric* of $\mathcal{O}$. If $C \in \{\mathbb{D}, \hat{\mathbb{C}}\}$, this metric is uniquely determined by normalizing the curvature to ± 1, and for $C = \mathbb{C}$ the metric is well-defined up to a positive scalar multiple. The orbifold metric on $\mathcal{O}$ determines a metric in the surface S with singularities at the ramified points of $\mathcal{O}$. More precisely, if $\nu(w_0) = m > 1$ for some $w_0 \in S$, then $\rho_{\mathcal{O}}(w)|dw|$ has a singularity of the type $|w - w_0|^{(1-m)/m}$ near w_0 in S. We then say that w_0 is a *cone point*.

Remark (Cone points versus punctures)*.* There is an advantage to defining an orbifold metric on $S \subsetneq \mathbb{C}$ for which w_0 is a cone point over inducing a hyperbolic metric in the punctured surface $S \setminus \{w_0\}$. Even if both of the corresponding densities tend to infinity as we approach w_0, contrary to what happens when w_0 is a puncture, the orbifold distance from a point of S to the cone point w_0 is finite, since w_0 is part of the surface. See [Mil06, pages 210-211], as well as Proposition 3.25 for an example where estimates are computed.

Remark (Metrics equivalence)*.* If $\mathcal{O} = (S, \nu)$, with $S \subset \mathbb{C}$, is an orbifold that admits an orbifold metric in the sense above, then the corresponding induced metric in S is topologically equivalent to the Euclidean metric in S. That is, both metrics generate the same topology on S. We will use this fact without further comment.

Let $\mathcal{O} = (S, \nu)$ be an orbifold, with $S \subsetneq \mathbb{C}$, that admits an orbifold metric $\rho_{\mathcal{O}}(w)|dw|$. This metric induces an $\mathcal{O}$-distance $d_{\mathcal{O}}(x, y)$ between points $x, y \in S$ in the following way. We join x to y by a rectifiable curve γ in S, and define the $\mathcal{O}$-length $\ell_{\mathcal{O}}(\gamma)$ of γ by

$$\ell_{\mathcal{O}}(\gamma) := \int_{\gamma} \rho_{\mathcal{O}}(w)|dw|.$$

Note that the integral is well-defined, since the set of ramified points in γ, and thus singularities of $\rho_{\mathcal{O}}$, is finite. See for example [BM17, A.1 and A.10] for more details on conformal and orbifold metrics. Finally, we set

$$d_{\mathcal{O}}(x, y) := \inf\{\ell_{\mathcal{O}}(\gamma) : \gamma \text{ is a rectifiable curve in } S \text{ joining } x \text{ and } y\}.$$

In particular, for any two subsets $A, B \subset S$, which may be singletons, we denote

$$d_{\mathcal{O}}(A, B) := \inf\{d_{\mathcal{O}}(x, y) : x \in A, y \in B\}.$$

The so-called Schwarz Lemma or Pick's theorem for hyperbolic surfaces [BM07, Theorem 6.4] generalizes to hyperbolic orbifolds in the following:

Theorem 2.6 (Orbifold Pick's theorem)**.** *A holomorphic map between two hyperbolic orbifolds can never increase distances as measured in the hyperbolic orbifold* *metrics. Distances are strictly decreased, unless the map is a covering map, in which case it is a local isometry.*

See [Thu84b, Proposition 17.4] or [McM94, Theorem A.3] for more details. Recall that as a consequence of Pick's theorem for hyperbolic surfaces, if U and V are hyperbolic domains with $V \subset U$, the inclusion from V into U is contracting, and so, it holds for their hyperbolic densities ρ_U and ρ_V that $\rho_U(z) \geq \rho_V(z)$ for all $z \in V$. Theorem 2.6 has analogous implications, that we shall use.

Remark. From now on, we use the notation $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ to indicate the inclusion map between $\widetilde{\mathcal{O}}$ and $\mathcal{O}$. We are implicitly stating that such map is well-defined, and in particular, the underlying surface of $\widetilde{\mathcal{O}}$ is contained in the underlying surface of $\mathcal{O}$.

Corollary 2.7 (Comparison of orbifold densities)**.** *Let $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ be hyperbolic orbifolds for which the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic. If $\rho_{\widetilde{\mathcal{O}}}$ and $\rho_{\mathcal{O}}$ are the respective densities of their orbifold metrics, then $\rho_{\widetilde{\mathcal{O}}}(z) \geq \rho_{\mathcal{O}}(z)$ for all unramified $z \in \widetilde{\mathcal{O}}$, with strict inequality when the inclusion map is not an orbifold covering.*

Observation 2.8 (Relative densities)**.** Recall that if $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, then all ramified points of $\mathcal{O}$ are also ramified points of $\widetilde{\mathcal{O}}$, and so the quotient $\rho_{\widetilde{\mathcal{O}}}(z)/\rho_{\mathcal{O}}(z)$ is well-defined for all unramified $z \in \widetilde{\mathcal{O}}$. Moreover, if $f\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is a covering map, then by Theorem 2.6 it is a local isometry, and therefore, $\rho_{\widetilde{\mathcal{O}}}(z) = |f'(z)|\rho_{\mathcal{O}}(f(z))$ for all $z \in \widetilde{\mathcal{O}} \cap \mathcal{O}$.

By the previous observation and Corollary 2.7, the following holds:

Corollary 2.9 (Lower bound on hyperbolic derivative)**.** *Let $f\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ be a covering map between hyperbolic orbifolds for which the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic but not a covering. Let $\rho_{\widetilde{\mathcal{O}}}$ and $\rho_{\mathcal{O}}$ be the respective densities of their orbifold metrics. Then, for all unramified $z \in \widetilde{\mathcal{O}}$,*

$$\|Df(z)\|_{\mathcal{O}} = \frac{|f'(z)|\rho_{\mathcal{O}}(f(z))}{\rho_{\mathcal{O}}(z)} = \frac{\rho_{\widetilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} > 1. \tag{2.2.3}$$

2.3 Background on holomorphic dynamics

We start by providing the most essential definitions and results on iteration of holomorphic maps necessary for the understanding of this thesis. For general texts in holomorphic dynamics we refer to [Mil06, Bea91], and more specifically, for iteration of meromorphic and transcendental functions, to [Ber93, Sch10]. We note that the results in this thesis concern mostly functions in the Eremenko-Lyubich class $\mathcal{B}$, and so we recommend [Six18b] for a survey of results. Unless otherwise stated, in this section $f\colon \mathbb{C} \to \mathbb{C}$ is a non-constant non-linear entire map. Recall that for each $n \geq 1$, f^n denotes the n-th iterate of f.

Definition 2.10 (Periodic points and cycles)**.** A point $z \in \mathbb{C}$ is a *periodic* point of f if there exists an integer $n \geq 1$ such that $f^n(z) = z$. The smallest n with this property is called the *period* of z. A periodic point of period one is a *fixed point*. A point $z \in \mathbb{C}$ is *preperiodic* if $f^n(z)$ is a periodic point for some $n \geq 1$, and we say that z is *strictly preperiodic* if it is preperiodic but not periodic. If z is a periodic point of f, then $\mathrm{Orb}^+(z)$ is a *periodic cycle*. The *multiplier* of a periodic point z of period n is $\mu(z) := (f^n)'(z)$. A periodic point z is called *attracting* if $0 \leq |\mu(z)| < 1$, *indifferent* if $|\mu(z)| = 1$ and *repelling* if $|\mu(z)| > 1$. An attracting periodic point z is *superattracting* if $\mu(z) = 0$. Since the multiplier of an indifferent periodic point is of the from $e^{2\pi i t}$ for some $0 \leq t < 1$, we can distinguish between *rationally* and *irrationally indifferent* points according to whether t is rational or not. A rationally indifferent periodic point is also called *parabolic*.

We denote by $\mathcal{A}(f)$ the set of all points whose forward orbit converges to some attracting cycle of f. The following property will be of use to us when $f \in \mathcal{B}$ is postcritically separated, since as we shall see in Lemma 3.5, in that case $P_F \Subset F(f) = \mathcal{A}(f)$. By *Jordan domain* we mean a complementary component of a Jordan curve on the sphere that is also a simply connected domain in $\mathbb{C}$. In particular, it might be bounded or unbounded.

Proposition 2.11 (Compact subsets of attracting basins [MB09, Proposition 3.1])**.** *Let f be a transcendental entire function and let $C \subset \mathcal{A}(f)$ be a compact set. Then there exist bounded Jordan domains $U_1, \dots, U_n$ compactly contained in pairwise different components of $\mathcal{A}(f)$ such that if $U := \bigcup_{i=1}^{n} U_i$, then*

$$f(U) \Subset U \Subset \mathcal{A}(f) \quad \text{and} \quad Orb^+(C) \Subset U.$$

In Chapter 1 we defined the set of singular values for a holomorphic function as the set of singularities of its inverse. Here we provide an equivalent characterization in terms of asymptotic and critical values, see for example [GK86, Lemma 1.1] for a proof of this equivalence.

Definition 2.12 (Singular values)**.** A *critical value* of f is the image of a point c for which $f'(c) = 0$; such a point c is called a *critical point*. Recall that the set of all critical values of f is denoted by $\mathrm{CV}(f)$ and that of critical points by $\mathrm{Crit}(f)$. A *(finite) asymptotic value* of f is a point $a \in \mathbb{C}$ for which there exists a curve $\gamma\colon (0, \infty) \to \mathbb{C}$ with $\gamma(t) \xrightarrow{t\to\infty} \infty$ such that $\lim_{t\to\infty} f(\gamma(t)) = a$. We write $\mathrm{AV}(f)$ for the set of all asymptotic values of f. The *singular set* of f is $S(f) := \overline{\{\mathrm{AV}(f) \cup \mathrm{CV}(f)\}}$ and its *postsingular set* $P(f) := \overline{\bigcup_{n\geq 0} f^n(S(f))}$. If $z \in S(f)$ then z is a *singular value*, otherwise z is a *regular* point.

We shall use later on that the singular set of every iterate of f is contained in its postsingular set:

Proposition 2.13 (Singular values of iterates are postsingular points)**.** *If f is an entire function, then for any $k \geq 1$,*

$$S(f^k) = \bigcup_{j=0}^{k-1} \overline{f^j(S(f))} \quad \subseteq \quad P(f). \tag{2.3.1}$$

Proof. We will prove that for any two entire functions f, g,

$$S(g \circ f) = S(g) \cup \overline{g(S(f))}, \tag{2.3.2}$$

and the statement will follow by induction on $k \geq 1$. We start by showing that

$$\mathrm{AV}(g \circ f) = \mathrm{AV}(g) \cup g(\mathrm{AV}(f)). \tag{2.3.3}$$

If $a \in \mathrm{AV}(g)$, then we can choose $\gamma \subset f(\mathbb{C})$ with $\gamma(t) \to \infty$ and $g(\gamma(t)) \to a$ as $t \to \infty$. Let $\tilde{\gamma}$ be a preimage under f of γ, which is an unbounded curve such that $f(\tilde{\gamma}) = \gamma$. Then, $(g \circ f)(\tilde{\gamma}(t)) \to a$ as $t \to \infty$, and so $a \in \mathrm{AV}(g \circ f)$. If $b \in g(\mathrm{AV}(f))$, then there exists $a \in \mathrm{AV}(f)$ such that $g(a) = b$, and so there exists a curve γ_a with $\gamma_a(t) \to \infty$ such that $f(\gamma_a(t)) \xrightarrow{t\to\infty} a$ and $(g \circ f)(\gamma_a(t)) \xrightarrow{t\to\infty} b$, and hence $b \in \mathrm{AV}(g \circ f)$. If $a \in \mathrm{AV}(g \circ f)$, there exists γ such $(g \circ f)(\gamma(t)) \to a$ as $t \to \infty$. Then, either $f(\gamma(t)) \to \tilde{a}$ for some $\tilde{a} \in f(\mathbb{C}) \cap g^{-1}(a)$, and so $a \in g(\mathrm{AV}(f))$, or $\tilde{\gamma} := f(\gamma)$ is unbounded curve, and so $a \in \mathrm{AV}(g)$. Thus, we have shown (2.3.3). By the chain rule,

$$g(f(z))' = 0 \Leftrightarrow z \in \mathrm{Crit}(f) \text{ or } f(z) \in \mathrm{Crit}(g) \Leftrightarrow f(z) \in \mathrm{CV}(f) \text{ or } g(f(z)) \in \mathrm{CV}(g),$$

and hence $\mathrm{CV}(g \circ f) = \mathrm{CV}(g_{|f(\mathbb{C})}) \cup g(\mathrm{CV}(f))$. Note that if $a \in \mathrm{CV}(g) \setminus f(\mathbb{C})$, then $a \in \mathrm{AV}(f)$ (see for example [Sch10, Theorem 1.14]), and therefore it holds $\mathrm{CV}(g) \subseteq \mathrm{CV}(g_{|f(\mathbb{C})}) \cup g(\mathrm{AV}(f))$. Using this and that the closure of a finite union of sets equals the union of their closures,

$$S(g \circ f) = \overline{\{\mathrm{AV}(g \circ f) \cup \mathrm{CV}(g \circ f)\}} = S(g) \cup \overline{g(\mathrm{AV}(f)) \cup \mathrm{CV}(f))}. \tag{2.3.4}$$

For a continuous map f and any subset S of its domain, $\overline{f(\overline{S})} = \overline{f(S)}$ and therefore, (2.3.2) follows from (2.3.4). $\square$

Recall from Chapter 1 that the *Fatou set* $F(f)$ of f is the largest set where the family $\{f^n\}_{n\geq 0}$ is normal, and its *Julia set* is $J(f) := \mathbb{C} \setminus F(f)$. We note that these sets are completely invariant under the action of f [Ber93, Lemma 2], and in particular, every iterate of a connected component of $F(f)$ is contained in a component of $F(f)$. This allows us to classify these connected components in terms of their orbits, in a similar fashion as we did for points:

Definition 2.14 (Fatou components)**.** A *Fatou component* U of f is a connected component of $F(f)$. U is *periodic* if there exists $n \geq 1$ such that $f^n(U) \subseteq U$. We call U *preperiodic* if $f^n(U)$ is periodic for some $n > 1$. A component which is preperiodic but not periodic is called *strictly preperiodic*. If U is not preperiodic, then U is a *wandering domain*, and these can be classified into three types: a wandering domain U is *escaping* if $U \subseteq I(f)$, *bounded* if the orbits of all points in U are bounded, or *(orbitally) oscillating* if it is neither bounded nor escaping.

The classification of wandering domains provided in Definition 2.14 is possible because if U is a wandering domain, then all limit functions of any convergent subsequence $\{f^{n_k}|_U\}_{k\geq 0}$ are constant and they equal either infinity or a finite limit point of $P(f)$. See Proposition 2.16(3). Periodic Fatou components can further be classified according to the dynamics occurring within them. Even if a classification is well-understood for functions as general as meromorphic [Ber93, Theorem 6], we state the classification for transcendental entire functions, that will be the only one relevant to us.

Theorem 2.15 (Classification of periodic Fatou components)**.** *Let f be a transcendental entire function and let U be a periodic Fatou component of period p. Then one of the following holds:*

- *U contains an attracting periodic point z_0 of period p. Then $f^n(z) \xrightarrow{n\to\infty} z_0$ for all $z \in U$, and U is called the* immediate attracting basin *of z_0.*
- *∂U contains a periodic point z_0 of period p and $f^n(z) \xrightarrow{n\to\infty} z_0$ for all $z \in U$. Then $(f^p)(z_0) = 1$ and U is a* Leau domain*, also called* immediate parabolic basin.
- *There exists an analytic homeomorphism $\Phi\colon U \to \mathbb{D}$ such that it holds $\Phi(f^p(\Phi^{-1}(z))) = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then, U is called a* Siegel disk.
- *$f^n(z) \xrightarrow{n\to\infty} \infty$ for all $z \in U$. In this case, U is called a* Baker domain.

Next, we gather together some results that relate singular values and different types of Fatou components, that we shall use in this thesis. We denote by $P(f)'$ the *derived set* of $P(f)$, that is, the set of its finite limit points.

Theorem 2.16 (Singular values and Fatou components)**.** *Let f be a transcendental entire function. Then the following hold:*

(1) Every cycle of immediate attracting or parabolic basins contains a singular value.

(2) Every boundary point of each Siegel disk is a limit point of $P(f)$.

(3) If U is a wandering domain of f, then all limit functions of $\{f^n|_U\}_{n\geq 0}$ are constant and contained in $P(f)' \cup \{\infty\}$.

Proof. For (1) see for example [Ber93, Theorem 7]. For Siegel disks, item (2) is due to Fatou. Even if stated only for rational maps, its proof applies also to transcendental ones. See also [Mil06, Corollary 14.4]. Item (3) is the main result in [BHK+93]. □

Finally, we turn our attention to the *escaping set*

$$I(f) := \left\{ z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty \right\}$$

of transcendental entire maps. Eremenko [Erë89] performed the first systematic study of this set for all entire transcendental maps, and subsequently, together with Lyubich [EL92], for those with bounded singular set. In particular, we shall frequently use the following relations:

Theorem 2.17 (Properties of the escaping set [Erë89, EL92])**.** *If f is a transcendental entire map, then $I(f) \neq \emptyset$, $J(f) = \partial I(f)$ and $I(f) \cap J(f) \neq \emptyset$. If in addition $f \in \mathcal{B}$, then $I(f) \cap F(f) = \emptyset$, and so in this case $J(f) = \overline{I(f)}$.*

2.4 External addresses and symbolic dynamics

The concept of *external address* for functions in class $\mathcal{B}$ allows to assign symbolic dynamics to points whose orbit stays away from a neighbourhood of their singular set. In this section, we review its definition and study properties of the sets of points sharing a same external address, with special emphasis on functions that are also criniferous. In particular, we show in Theorem 2.30 that for criniferous functions, these sets are ray tails or landing dynamic rays. We start by partitioning a subset of the plane where f "stays large":

Definition 2.18 (Tracts, fundamental domains)**.** Fix $f \in \mathcal{B}$ and let D be a bounded Jordan domain around the origin containing $S(f)$ and $f(0)$. Each connected component of $f^{-1}(\mathbb{C} \setminus \overline{D})$ is a *tract* of f, and $\mathcal{T}_f$ denotes the set of all tracts. Let δ be an arc connecting a point of $\overline{D}$ to infinity in the complement of the closure of the tracts. Denote

$$\mathcal{W} := \mathbb{C} \setminus (\overline{D} \cup \delta). \tag{2.4.1}$$

Each connected component of $f^{-1}(\mathcal{W})$ is a *fundamental domain* of f.

Remark. Sometimes throughout the document, $\mathcal{T}_f$ will stand for both the set of tracts of f and for the subset of $\mathbb{C}$ that equals the union of the tracts of f. Most of the times, this differentiation is not important and the meaning will be clear from the context.

The following are well-known facts about the concepts just defined:

Proposition 2.19 (Properties of tracts and fundamental domains)**.** *In the setting of Definition 2.18, the following hold:*

(1) Each tract $T \in \mathcal{T}_f$ is an unbounded Jordan domain.

(2) The restriction of f to the closure of each $T \in \mathcal{T}_f$, that we denote $f|_{\overline{T}}$, is a universal covering map of infinite degree.

(3) Fundamental domains are well-defined, and for any fundamental domain F, $f|_F : F \to \mathcal{W}$ is a conformal isomorphism.

(4) Only finitely many tracts or fundamental domains can intersect any given compact set K. In fact, $\mathcal{T}_f \cap K$ has only finitely many components. In particular, at most finitely many closures of fundamental domains intersect $\overline{D}$.

Proof. Since $S(f) \subset D$, $f|_T$ is a covering map, and because f is transcendental, by the classification of covering maps of the punctured disc [For99, Theorem 5.10], the tract $T \in \mathcal{T}_f$ is simply-connected and unbounded, and $f : T \to \mathbb{C} \setminus \overline{D}$ is a universal covering map of infinite degree. By taking a slightly smaller domain $D \Supset \tilde{D} \Supset S(f)$ and applying these observations to $\tilde{D}$ and the new collection of tracts $\mathcal{T}_{\tilde{D}}$, ∂T becomes the preimage of the simple closed curve ∂D under a universal covering map, and $f|_{\partial T}$ is a covering of ∂D. Consequently, (1) and (2) follow. Note that by the assumption on $D \supset \{0, f(0)\}$, $0 \notin \mathcal{T}_f$ and so $\overline{D} \nsubseteq \mathcal{T}_f$. Then, the existence of the curve $\delta \subset \mathbb{C} \setminus (\overline{D} \cup \overline{\mathcal{T}}_f)$ connecting ∂D to ∞ can be seen noting that $\widehat{\mathbb{C}} \setminus (\overline{D} \cup \overline{\mathcal{T}}_f)$ is open and that the boundaries of its connected components are locally connected: if U is the connected component containing infinity, by applying Carathéodory-Torhorst's Theorem[2] [Pom92, Theorem 2.1]

[2]This theorem is commonly attributed only to Carathéodory, although its first full proof seems to date back to 1921 and was given by Marie Torhorst [Tor21]. See [RG14, §2] for further discussion.

to extend the Riemann map $\varphi\colon U \to \mathbb{D}$ to the continuous map $\tilde{\varphi}\colon \overline{U} \to \overline{\mathbb{D}}$, since any two points in $\overline{\mathbb{D}}$ can be connected by a curve in $\overline{\mathbb{D}}$, (3) follows. For the proof of a more general version of (4), see [BRG17, Lemma 2.1]. □

Recall that $f \in \mathcal{B}$ is of *disjoint type* if it is hyperbolic and its Fatou set is connected. Alternatively, disjoint type maps can be characterized as those maps for which tracts can be defined so that their boundaries are *disjoint* from the boundary of their image. More precisely:

Proposition 2.20 (Characterization of disjoint type maps)**.** *A function $f \in \mathcal{B}$ is of disjoint type if and only if there exists a Jordan domain $D \supset S(f)$ such that $\overline{f(D)} \subset \overline{D}$.*

See for example [MB12, Proposition 2.8] for a proof of Proposition 2.20. The partition of $f^{-1}(\mathcal{W})$ into fundamental domains allows us to assign symbolic dynamics to those points whose orbit stays in $\mathcal{W}$. For each fundamental domain F of f, $\overset{\infty}{F}$ denotes the unbounded connected component of $F \setminus \overline{D}$.

Definition 2.21 (External addresses for functions in class $\mathcal{B}$)**.** Let $f \in \mathcal{B}$ and suppose that fundamental domains have been defined for f. An *(infinite) external address* is a sequence $\underline{s} = F_0F_1F_2\dots$ of fundamental domains of f. The external address $\underline{s}$ is *bounded* if the set of fundamental domains occurring in $\underline{s}$ is finite. For each external address $\underline{s}$, we denote

$$J_{\underline{s}} := \left\{ z \in \mathbb{C}\colon f^n(z) \in \overset{\infty}{F}_n \text{ for all } n \geq 0 \right\}. \tag{2.4.2}$$

We say that $\underline{s}$ is *admissible* if $J_{\underline{s}}$ is non-empty, and we denote by $\operatorname{Addr}(f)$ the set of all admissible external addresses. If $z \in J_{\underline{s}}$ for some $\underline{s} \in \operatorname{Addr}(f)$, then we say that z *has (external) address* $\underline{s}$. Moreover, σ stands for the one-sided *shift operator* on external addresses. That is, $\sigma(F_0F_1F_2\dots) = F_1F_2\dots$. In particular,

$$f(J_{\underline{s}}) \subseteq J_{\sigma(\underline{s})} \qquad \text{for all} \qquad \underline{s} \in \operatorname{Addr}(f). \tag{2.4.3}$$

Notation. Let $\underline{s} = s_0s_1s_2\dots \in \operatorname{Addr}(f)$ and suppose that there exists $N \geq 0$ such that $s_i = s_N$ for all $i \geq N$. Then we write $\underline{s} = s_0\dots s_{N-1}\overline{s_N}$.

Remark. For the reader familiar with [BRG17], we note that the sets "$J_{\underline{s}}$" are denoted by "$J^0_{\underline{s}}(f)$" in [BRG17, Definition 2.4], and do not equal the sets "$J_{\underline{s}}$" introduced in [BRG17, Definition 4.2]. We have waived consistency in notation across articles in favour of simplifying notation in ours. Moreover, we remark that the choice of the letter "J" for the sets in (2.4.2) is not arbitrary: they lie entirely in the Julia set of f, see [BRG17, Lemma 2.6]. Thus, we informally refer to the sets "$J_{\underline{s}}$" as *Julia constituents.*

Observation 2.22 (Points with external address)**.** Whenever it is defined, the external address of a point is unique because Julia constituents are by definition pairwise disjoint. If f is of disjoint type, then by Proposition 2.20, there exists a choice of tracts and fundamental domains so that $\overset{\infty}{F} = F$ for each fundamental domain F. This implies that the Julia set of f can be described as the disjoint union of its Julia constituents, that is,

$$f \text{ is of disjoint type} \quad \Rightarrow \quad J(f) = \bigcup_{\underline{s} \in \operatorname{Addr}(f)} J_{\underline{s}}, \tag{2.4.4}$$

and in particular all points in $J(f)$ have an external address. However, this is not the case for all functions in class $\mathcal{B}$, as for example occurs when $S(f) \cap J(f)$ is not empty.

For the rest of the section, we assume for each $f \in \mathcal{B}$ that $\operatorname{Addr}(f)$ has been defined for some choice of fundamental domains and tracts $\mathcal{T}_f$. We will use in this section the following properties of Julia constituents.

Theorem 2.23 (Realisation of addresses [BRG17, Theorem 2.5])**.** *Let $f \in \mathcal{B}$. Then, for each external address $\underline{s}$, the following holds.*

(a) If $\underline{s}$ is admissible, then $J_{\underline{s}}$ contains a closed, unbounded, connected set X on which the iterates of f tend to infinity uniformly.

(b) If X_1 and X_2 are unbounded, closed, connected subsets of $J_{\underline{s}}$ with $X_1 \not\subseteq X_2$, then $X_2 \subseteq X_1$ and $f^n|_{X_2} \to \infty$ uniformly.

(c) If $\underline{s}$ is bounded, then it is admissible, that is, $J_{\underline{s}} \neq \emptyset$.

Note that Julia constituents need not be connected nor closed. Thus, we shall usually and instead work with the following subsets:

2.24 (Closed sets in Julia constituents)**.** For each $\underline{s} \in \operatorname{Addr}(f)$, we denote by $J^{\infty}_{\underline{s}}$ the closure of the union of all closed, unbounded, connected sets $X \subset J_{\underline{s}}$ on which the iterates of f tend to infinity uniformly.

Before we continue the study of Julia constituents, we note that an advantage that functions in class $\mathcal{B}$ present over other transcendental entire maps is that their restriction to each of their tracts is a covering map that *expands uniformly* the hyperbolic metric that sits on $\mathbb{C} \setminus \overline{D}$, where $D \supset S(f)$ is the bounded set chosen to define tracts. This well-known fact lies behind many results for functions in this class, and goes back to [EL92, Lemma 1]. We denote by $\rho_{\mathbb{C}\setminus\overline{D}}$ the density of the hyperbolic metric in $\mathbb{C} \setminus \overline{D}$.

Proposition 2.25 (Hyperbolic expansion on tracts)**.** *Let $f \in \mathcal{B}$, fix a domain $D \supset S(f)$ and let $\mathcal{T}_f := f^{-1}(D)$. For each tract $T \in \mathcal{T}_f$, denote by $\overset{\infty}{T}$ the unbounded connected component of $T \setminus \overline{D}$, and let $\overset{\infty}{\mathcal{T}} := \bigcup_{T\in\mathcal{T}_f} \overset{\infty}{T}$. Then, there exists a constant $\Delta > 1$, depending only on ∂D and f, such that*

$$\|Df(z)\|_{\mathbb{C}\setminus\overline{D}} := |f'(z)| \cdot \frac{\rho_{\mathbb{C}\setminus\overline{D}}(f(z))}{\rho_{\mathbb{C}\setminus\overline{D}}(z)} > \Delta$$

for all $z \in \overset{\infty}{\mathcal{T}}$. Moreover, let $\{F_i\}_{i\in I}$ be a choice of fundamental domains of f defined with respect to a domain D and curve δ. If $S(f) \Subset D' \Subset D$ for a subdomain D', then for each $R > 0$, there exists $C > 0$ such that for all $i \in I$, the Euclidean and hyperbolic diameter of $\overset{\infty}{F_i} \cap \mathbb{D}_R$ is less than C, where the hyperbolic metric sits in $\mathbb{C} \setminus \overline{D}'$.

Proof. For a proof of the first part of the statement see for example [Rem09, Lemma 5.1]. For the second part, note that by Proposition 2.19(4), for each fixed $R > 0$, only finitely many fundamental domains intersect $\overline{\mathbb{D}}_R$. If F_i is one of those, then $\overset{\infty}{F_i} \cap \mathbb{D}_R$ is compactly contained in $\mathbb{C} \setminus \overline{D}'$, and so has finite Euclidean and hyperbolic diameter. □

In particular, this expansion has consequences on the Hausdorff dimension of the non-escaping points of the sets from 2.24. For each set $A \subset \mathbb{C}$, we denote its Hausdorff dimension by $\dim_H A$. See for example [Fal14] for definitions.

Proposition 2.26 (Hausdorff dimension of non-escaping points with same address)**.** *Let $f \in \mathcal{B}$. Then, for each $\underline{s} \in \operatorname{Addr}(f)$, the set of non-escaping points in $J^{\infty}_{\underline{s}}$ has Hausdorff dimension zero.*

Remark. The idea of the following proof is essentially the same as that for [RG16, proof of Proposition 5.9]. Still, for completeness, we include it with the minor modifications that adapt it to our setting.

Proof of Proposition 2.26. For each $\underline{s} \in \operatorname{Addr}(f)$ and $n \in \mathbb{N}_{\geq 1}$, we denote

$$f^n_{\underline{s}} := f|_{\overset{\infty}{F}_{n-1}} \circ f|_{\overset{\infty}{F}_{n-2}} \circ \cdots \circ f|_{\overset{\infty}{F}_0} \quad \text{and} \quad f^{-n}_{\underline{s}} := \left(f^n_{\underline{s}}\right)^{-1}.$$

If $z \notin I(f)$, then there is $K > 0$ such that $f^n(z) \in \mathbb{D}_K$ for infinitely many $n \geq 0$. Hence, if $\underline{s} = F_0F_1F_2\ldots$, then the set of non-escaping points in $J^{\infty}_{\underline{s}}$ can be written as

$$J^{\infty}_{\underline{s}} \setminus I(f) = \bigcup_{K=0}^{\infty} \bigcap_{n_0=0}^{\infty} \bigcup_{n=n_0}^{\infty} f^{-n}_{\underline{s}}(\overset{\infty}{F}_n \cap \mathbb{D}_K).$$

Since a countable union of sets of Hausdorff dimension zero has Hausdorff dimension zero, it is sufficient to prove that for each $K > 0$, the set

$$S(K) := \bigcap_{n_0=0}^{\infty} \bigcup_{n=n_0}^{\infty} f_{\underline{s}}^{-n}(\overset{\infty}{F}_n \cap \mathbb{D}_K)$$

has Hausdorff dimension zero. Let us fix some arbitrary $K > 0$. Suppose that fundamental domains have been defined with respect to a domain D and curve δ. Choose a subdomain D' such that $S(f) \Subset D' \Subset D$ and define two sets of tracts for f, as connected components of $f^{-1}(\mathbb{C} \setminus \overline{D})$ and $f^{-1}(\mathbb{C} \setminus \overline{D}')$, that we denote respectively by $\mathcal{T}_D$ and $\mathcal{T}_{D'}$. In particular, $\mathcal{T}_D \Subset \mathcal{T}_{D'}$. Then, by Proposition 2.25, there exists a constant $C := C(K)$ such that for all n, the set of points in $\overset{\infty}{F}_n \cap \overline{\mathbb{D}}_K$ has diameter at most C in the hyperbolic metric of $\mathbb{C} \setminus \overline{D}'$. Moreover, by the same proposition, there exists a constant $\varDelta > 1$ such that $\|Df(z)\|_{\mathbb{C}\setminus\overline{D}'} \geq 1$. In particular, for any domain $S \subset \mathbb{C} \setminus \overline{D}'$ such that $f^{-1}(S) \subset \mathbb{C} \setminus \overline{D}'$, $\operatorname{diam}_{\mathbb{C}\setminus\overline{D}'}(f^{-1}(S)) \leq \operatorname{diam}_{\mathbb{C}\setminus\overline{D}'}(S) \cdot \varDelta$, where $\operatorname{diam}_{\mathbb{C}\setminus\overline{D}'}$ denotes the hyperbolic diameter in $\mathbb{C} \setminus \overline{D}'$. Let us assume that for each fundamental domain F, the subset $\overset{\infty}{F}$ is endowed with a hyperbolic metric. Then, since for each $n \geq 1$ the restriction $f|_{\overset{\infty}{F}_n}$ is a hyperbolic isometry to a subset of $\mathbb{C} \setminus \overline{D}'$ by Schwarz-Pick Lemma [BM07, Lemma 6.4], using the observation above,

$$\text{for} \quad S_n := f_{\underline{s}}^{-n}(\overset{\infty}{F}_n \cap \mathbb{D}_K), \quad \operatorname{diam}_{\overset{\infty}{F}_0}(S_n) \leq C \cdot \varDelta^{-(n-1)}, \tag{2.4.5}$$

where $\operatorname{diam}_{\overset{\infty}{F}_0}$ denotes hyperbolic diameter in $\overset{\infty}{F}_0$. Since $0 \notin \mathcal{T}_f$, for each $R \in \mathbb{N}_{\geq 1}$, the Euclidean distance between any point $z \in \overset{\infty}{F}_0 \cap \mathbb{D}_R$ and $\partial \overset{\infty}{F}_0$ is at most $2R$. Thus, by a standard estimate on the hyperbolic metric in a simply-connected domain [BM07, Theorem 8.6], $\rho_{\overset{\infty}{F}_0}(z) \geq 1/(4R)$ for all $z \in \overset{\infty}{F}_0 \cap \mathbb{D}_R$. Hence, by (2.4.5), the Euclidean diameter of $S_n \cap \mathbb{D}_R$ is at most $4R \cdot C \cdot \varDelta^{-(n-1)}$. Then, for a fixed $t > 0$ and for every $n_0 \geq 1$, the t-dimensional Hausdorff measure of $S(K) \cap \mathbb{D}_R$ is bounded from above by

$$\begin{aligned}\liminf_{n_0\to\infty} \sum_{n \geq n_0} \operatorname{diam}(S_n \cap \mathbb{D}_R)^t &\leq \liminf_{n_0\to\infty} \sum_{n\geq n_0} (4R \cdot C \cdot \varDelta^{(-(n-1))})^t \\ &= (4RC)^t \cdot \lim_{n_0\to\infty} \sum_{n \geq n_0 - 1} (\varDelta^{-t})^n = 0.\end{aligned}$$

Thus, $\dim_H(S(K) \cap \mathbb{D}_R) \leq t$. Since $t > 0$ was arbitrary, $\dim_H(S(K) \cap \mathbb{D}_R) = 0$. Using again that a countable union of Hausdorff dimension zero has Hausdorff dimension zero, $\dim_H(S(K)) = 0$. □

We shall next see in Theorem 2.30 that for criniferous functions, the sets from

2.24 are either ray tails or dynamic rays together with their endpoints. We first recall the definition of criniferous functions introduced in [BRG17, Definition 1.2].

Definition 2.27 (Criniferous functions)**.** An entire function f is *criniferous* if every $z \in I(f)$ is eventually mapped to a ray tail. That is, for every $z \in I(f)$, there exists a natural number $N := N(z)$ such that $f^n(z)$ belongs to a ray tail for all $n \geq N$.

In the proof of Theorem 2.30 we will use some results from continuum theory, and so we include some basic definitions and results. Recall that a *continuum* X (i.e., a non-empty compact, connected metric space) is called *indecomposable* if it cannot be written as the union of two proper subcontinua of X. The *composant of a point* $x \in X$ is the union of all proper subcontinua of X containing x, and a *composant* of X is a maximal set in which any two points lie within some proper subcontinuum of X. If X is indecomposable, then there are uncountably many different composants, every two of which are disjoint, and each of which is connected and dense in X, see [Nad92, Exercise 5.20(a) and Theorem 11.15].

Theorem 2.28 (Boundary bumping theorem [Nad92, Theorem 5.6])**.** *Let X be a continuum and let $E \subsetneq X$ be non-empty. If K is a connected component of $X \setminus E$, then $\overline{K} \cap \partial E \neq \emptyset$.*

We will moreover make use of the following result in order to show that the accumulation set of a dynamic ray is an indecomposable continuum:

Theorem 2.29 (Curry [Cur91])**.** *Suppose that γ is a ray, i.e. a continuous injective image of $[0,1)$, and let $\Lambda(\gamma)$ denote its accumulation set. If $\Lambda(\gamma)$ has topological dimension one, does not separate the Riemann sphere into infinitely many components and contains γ, then $\Lambda(\gamma)$ is an indecomposable continuum.*

Theorem 2.30 (Criniferous functions in $\mathcal{B}$)**.** *If $f \in \mathcal{B}$ is criniferous, then for each $\underline{s} \in \operatorname{Addr}(f)$, the set $J^\infty_{\underline{s}} \subseteq J_{\underline{s}}$ from 2.24 is either a ray tail or a dynamic ray together with its endpoint. In particular,*

$$I(f) \subset \bigcup_{n \geq 0} f^{-n}\Bigg(\bigcup_{\underline{s} \in \operatorname{Addr}(f)} J^\infty_{\underline{s}} \Bigg). \tag{2.4.6}$$

Proof. Fix $\underline{s} \in \operatorname{Addr}(f)$ and let us choose any $z \in J^\infty_{\underline{s}} \cap I(f)$. Since f is criniferous, there exists $N \geq 0$ so that $f^N(z)$ is the endpoint of a ray tail γ. Then, since by definition ray tails escape uniformly to infinity, there exists a constant $M := M(\gamma)$ such that $f^m(\gamma)$ is contained in the tracts of f for all $m \geq M$, which in particular implies that $f^m(\gamma)$ must be totally contained in a fundamental domain for each $m \geq M$. More specifically, since $f^{N+m}(z) \in f^m(\gamma)$, the curve $f^m(\gamma)$ belongs to the same fundamental domain $f^{N+m}(z)$ does, which in turn is determined by the external address $\underline{s}$. Hence, all points in $f^m(\gamma)$ have external address $\sigma^{m+N}(\underline{s})$, and in particular, $f^M(\gamma) \subset J^{\infty}_{\sigma^{N+M}(\underline{s})}$. Moreover, by (2.4.3), it also holds that $f^{N+M}(J^{\infty}_{\underline{s}}) \subseteq J^{\infty}_{\sigma^{N+M}(\underline{s})}$. Since the restriction of f to any Julia constituent is injective, as they all lie outside a Jordan domain that contains $S(f)$, by definition of $J^{\infty}_{\underline{s}}$ it must occur that $f^M(\gamma) \subseteq f^{N+M}(J^{\infty}_{\underline{s}})$. Hence, the curve in the $(N+M)$-th preimage of $f^M(\gamma)$ that intersects $J^{\infty}_{\underline{s}}$ must be a ray tail with endpoint z, that we denote by γ_z. That is, $\gamma_z := f^{-N-M}(f^M(\gamma)) \cap J^{\infty}_{\underline{s}}$.

If $z, w \in J^{\infty}_{\underline{s}} \cap I(f)$, then by Theorem 2.23(b) either $\gamma_w \subset \gamma_z$ or $\gamma_z \subset \gamma_w$, and thus, these curves are totally ordered by inclusion. Hence,

$$\gamma := \bigcup_{z \in I(f) \cap J^{\infty}_{\underline{s}}} \gamma_z$$

is a maximal injective curve in $I(f)$ that escapes uniformly to infinity, and in particular, $\gamma = I(f) \cap J^{\infty}_{\underline{s}}$. If $J^{\infty}_{\underline{s}} \subset I(f)$, then $\gamma = J^{\infty}_{\underline{s}}$ is a ray tail and we are done. Otherwise, let us parametrize $\gamma : (0, \infty) \to \mathbb{C}$, and denote by $\Lambda(\gamma)$ the accumulation set of $\gamma(t)$ as $t \to 0$. In particular, since $J^{\infty}_{\underline{s}}$ is closed, $J^{\infty}_{\underline{s}} \supseteq \gamma \cup \Lambda(\gamma)$. Let us compactify $J^{\infty}_{\underline{s}}$ by adding infinity, i.e. $\widehat{J}_{\underline{s}} := J^{\infty}_{\underline{s}} \cup \{\infty\}$. If $w \in J^{\infty}_{\underline{s}} \setminus \gamma$, then by the boundary bumping theorem (Theorem 2.28), if K is the connected component of $\widehat{J}_{\underline{s}} \setminus \gamma$ containing w, then $\overline{K} \cap \gamma \neq \emptyset$. But then, since $K \subset \widehat{J}_{\underline{s}} \setminus I(f)$, by Proposition 2.26, the set K must be the singleton $\{w\}$, and thus $w \in \Lambda(\gamma)$. Thus, we have shown that

$$J^{\infty}_{\underline{s}} = \gamma \cup \Lambda(\gamma).$$

Consequently, it suffices for our purposes to study the set $\Lambda(\gamma)$. First, we note that for every potential $t > 0$ of γ, there are pieces of other dynamic rays of f accumulating uniformly *from above* and *from below* on $\gamma_{[t,\infty)}$. This follows from a well-known argument, see for example the proof of [BRG17, Corollary 6.7] or [SZ03, Corollary 6.9], that we sketch here. Fix $z := \gamma(t^*)$ for some $t^* \geq t$. For each fundamental domain F, there exists a pair of fundamental domains F^- and F^+, that are respectively the immediate predecessor and successor of F in the cyclic order at infinity of fundamental domains, see 2.31. In particular, F^-, F, F^+ lie in the same tract. Then, for each $k \geq 0$, consider the external address $\underline{s}^+_k$ that equals $\underline{s}$ except on its k-th entry, which is F^+_k instead of F_k; and similarly, $\underline{s}^-_k$ equals $\underline{s}$ except that its k-th entry is F^-_k. Then, using the expansion property that f has in tracts, (Proposition 2.25), one can see by mapping forward and pulling back through appropriate inverse branches that $J_{\underline{s}_k^+}(t^*) \to z$ from above and $J_{\underline{s}_k^-}(t^*) \to z$ from below as $k \to \infty$.

This implies that if $\gamma(t_0) \in \Lambda(\gamma)$ for some $t_0 > 0$, then the curve γ must also accumulate on $\gamma([t, t_0])$ for all $< 0 < t < t_0$. Hence, by letting $t \to 0$ we see that $\gamma_{(0,t_0]} \subset \Lambda(\gamma)$. Let t_0 be any potential such that $\gamma(t_0) \in \Lambda(\gamma)$. Then, by Theorem 2.29, $\Lambda(\gamma_{(0,t_0]})$ must be an indecomposable continuum. Recall that this means that all composants of $\Lambda(\gamma_{(0,t_0]})$ are pairwise disjoint and dense in $\Lambda(\gamma_{(0,t_0]})$, and in particular, since their closures must contain $\gamma_{(0,t_0]}$, these composants must be non-trivial. However, this would contradict Proposition 2.26, and thus $\Lambda(\gamma)$ must be a singleton, namely the landing point of γ. Thus, $J^\infty_{\underline{s}} \setminus I(f)$ is the landing point of the dynamic ray γ.

For the second part of the statement, if $z \in I(f)$, then by the same argument as before, there exists $N := N(z) > 0$ and a ray tail γ such that $f^N(z) \in \gamma \subset J^\infty_{\underline{\tau}}$ for some $\underline{\tau} \in \operatorname{Addr}(f)$ and so (2.4.6) holds. $\square$

2.31 (Cyclic order and topology in $\operatorname{Addr}(f)$)**.** For any function $f \in \mathcal{B}$, there is a natural *cyclic order*[3] on the set of its fundamental domains (resp. the set of its Julia constituents): if A, B, C are fundamental domains (resp. Julia constituents), then we write

$$[A, B, C]_\infty \quad \Leftrightarrow \quad B \text{ tends to infinity between } A \text{ and } C \text{ in positive orientation.} \tag{2.4.7}$$

See [BRG17, Section 12] for details on the existence of a cyclic order on any pairwise disjoint collection of unbounded, closed, connected subsets of $\mathbb{C}$, none of which separates the plane. From this cyclic order, it is possible to define a *lexicographical cyclic order* on the set $\operatorname{Addr}(f)$: consider the cyclic order specified in (2.4.7) over the set of fundamental domains together with the curve δ used in (2.4.1). From this cyclic order, we can define a linear order on the set of fundamental domains by "cutting" δ the following way:

$$F < \tilde{F} \quad \text{if and only if} \quad [\delta, F, \tilde{F}]_\infty.$$

Then, the set of fundamental domains becomes a totally ordered set, and this order gives rise to a lexicographical order "$<_\ell$" on external addresses in the usual sense. In turn, $\operatorname{Addr}(f)$ becomes a totally ordered set, and hence we can define

[3] ternary relation $[A, B, C]$ that is cyclic, asymmetric, transitive and total.

a cyclic order induced by $<_\ell$ the usual way:

$$[\underline{s}, \underline{\alpha}, \underline{\tau}]_\ell \quad \text{if and only if} \quad \underline{s} <_\ell \underline{\alpha} <_\ell \underline{\tau} \quad \text{or} \quad \underline{\alpha} <_\ell \underline{\tau} <_\ell \underline{s} \quad \text{or} \quad \underline{\tau} <_\ell \underline{s} <_\ell \underline{\alpha}. \tag{2.4.8}$$

This cyclic order on addresses agrees with the cyclic order at infinity of their corresponding Julia constituents. That is,

$$[\underline{s}, \underline{\alpha}, \underline{\tau}]_\ell \quad \text{if and only if} \quad [J_{\underline{s}}, J_{\underline{\alpha}}, J_{\underline{\tau}}]_\infty. \tag{2.4.9}$$

The equivalence in (2.4.9) follows from the cyclicity axiom of ternary relations (that is, if $[a, b, c]$ then $[b, c, a]$) together with the following claim:

Claim. For any pair $\underline{s}, \underline{\alpha} \in \operatorname{Addr}(f)$ such that $\underline{s} <_\ell \underline{\alpha}$, it holds $[\delta, J_{\underline{s}}, J_{\underline{\alpha}}]_\infty$, where we have considered the cyclic order at infinity of all Julia constituents together with the curve δ.

Proof of claim. Suppose that $\underline{s} := F_0^s F_1^s \ldots$, and $\underline{\alpha} := F_0^\alpha F_1^\alpha \ldots$ and that $\underline{s}$ and $\underline{\alpha}$ first differ in their k-th entry. That is, $F_i^s = F_i^\alpha$ for all $i < k$, and $F_k^s \neq F_k^\alpha$. Note that $\underline{s} <_\ell \underline{\alpha}$ holds if and only if $f^i(J_{\underline{s}}), f^i(J_{\underline{\alpha}}) \subset F_i^s$ for all $i < k$ and $[\delta, f^k(J_{\underline{s}}), f^k(J_{\underline{\alpha}})]_\infty$. But then, since by Proposition 2.19(3) the function f acts as a conformal isomorphism from each fundamental domain to $\mathcal{W}$, in particular f preserves the cyclic order at infinity of Julia constituents. Thus,

$$\begin{aligned}[\delta, f^k(J_{\underline{s}}), f^k(J_{\underline{\alpha}})]_\infty &\Longleftrightarrow [\delta, f^{k-1}(J_{\underline{s}}), f^{k-1}(J_{\underline{\alpha}})]_\infty \Longleftrightarrow \cdots \Longleftrightarrow [\delta, f(J_{\underline{s}}), f(J_{\underline{\alpha}})]_\infty \\ &\Longleftrightarrow [\delta, J_{\underline{s}}, J_{\underline{\alpha}}]_\infty,\end{aligned}$$

and the claim follows. △

The cyclic order on addresses specified in (2.4.9) allows us to provide the set $\operatorname{Addr}(f)$ with a topology the following way: given two different elements $\underline{s}, \underline{\tau} \in \operatorname{Addr}(f)$, we define the *open interval* from $\underline{s}$ to $\underline{\tau}$, denoted by $(\underline{s}, \underline{\tau})$, as the set of all addresses $\underline{\alpha} \in \operatorname{Addr}(f)$ such that $[\underline{s}, \underline{\alpha}, \underline{\tau}]_\ell$. The collection of all such open intervals forms a base for the *cyclic order topology.* In particular, the open sets in this topology happen to be exactly those ones which are open in every compatible linear order.

Remark. Unless otherwise stated, from now on and when working with external addresses, we will assume that the set $\operatorname{Addr}(f)$ has been endowed with the cyclic order topology.

We would like to point out to the reader that providing $\operatorname{Addr}(f)$ with a topological structure allows us to use the notion of *convergence* of external addresses.

In particular, for disjoint type functions, convergence of addresses is closely related to how the corresponding Julia constituents *accumulate* on the plane. More specifically, let $f \in \mathcal{B}$ be of disjoint type, and for every $w \in J(f)$ denote by $\operatorname{add}(w)$ its external address. For a sequence of points $\{z_k\}_k$ in $J(f)$,

$$\text{if} \quad z_k \to z, \quad \text{then} \quad \operatorname{add}(z_k) \to \operatorname{add}(z) \quad \text{as } k \to \infty, \tag{2.4.10}$$

which is a consequence of the characterization of disjoint type functions in Proposition 2.20 and expansion from Proposition 2.25. See the proof of Theorem 2.30 for details on a similar argument.

2.5 Logarithmic coordinates

A commonly used tool for studying functions in the Eremenko-Lyubich class $\mathcal{B}$ is the *logarithmic change of coordinates*, a technique firstly used in this context in [EL92, Section 2]. Our incursion on the topic is very brief, since we will only make use of these coordinates to get some expansion estimates. For a more detailed overview of this technique we refer to [Six18b, Section 5], [RRRS11, Section 2] or [RG16, Section 3].

2.32 (Logarithmic transform)**.** For $f \in \mathcal{B}$, fix an Euclidean disk $\mathbb{D}_L \Supset S(f)$ and define the tracts $\mathcal{T}_f$ of f as the connected components of $f^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}_L)$. Let $\mathbb{H}_{\log L} := \exp^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}_L)$ be the right half plane containing all points with real part greater than $\log L$, and let $\mathcal{T}_F := \exp^{-1}(f^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}_L))$. Note that each connected component T of $\mathcal{T}_F$ is a simply connected domain whose boundary is homeomorphic to $\mathbb{R}$. Moreover, by the action of the exponential map, both "ends" of the boundaries of T have real parts converging to $+\infty$, and both $\mathcal{T}_F$ and $\mathbb{H}_{\log L}$ are invariant under translation by $2\pi i$. Consequently, we can lift f to a map $F : \mathcal{T}_F \to \mathbb{H}_{\log L}$ satisfying

$$\exp \circ F = f \circ \exp \tag{2.5.1}$$

and such that F is $2\pi i$-periodic. We call F a *logarithmic transform* of f. Moreover, we call each connected component of $\mathcal{T}_F$ a *logarithmic tract* of F.

By construction, the following facts, that will be useful for us, also hold:

2.33 (Properties of logarithmic transforms)**.** In the setting of 2.32, we have:

(1) Each tract $T \in \mathcal{T}_F$ is an unbounded Jordan domain that is disjoint from all its $2\pi i\mathbb{Z}$-translates. The restriction $F|_T \colon T \to \mathbb{H}_{\log L}$ is a conformal

isomorphism whose continuous extension to the closure of T in $\widehat{\mathbb{C}}$ satisfies $F(\infty) = \infty$. We denote the inverse of $F|_T$ by F_T^{-1}.

(2) The components of $\mathcal{T}_F$ have pairwise disjoint closures and accumulate only at ∞; i.e., if $\{z_n\}_n \subset \mathcal{T}_F$ is a sequence of points, all belonging to different components of $\mathcal{T}_F$, then $z_n \to \infty$ as $n \to \infty$.

Remark. It is possible to define functions satisfying the same properties that logarithmic transforms have that do not necessarily arise as a lift of some $f \in \mathcal{B}$. The class of such functions is denoted by $\mathcal{B}_{\log}$. See for example [RG16, Definition 3.3].

By Carathéodory-Torhorst's Theorem[4], for each $T \in \mathcal{T}_F$ the function $F|_T$ in 2.33(1) can be continuously extended to the boundary of T. In addition, since T is a Jordan domain, this extension is a homeomorphism, and in particular $F|_T$ extends continuously to a homeomorphism between the closures $\overline{T}$ and $\overline{\mathbb{H}}_{\log L}$ (taken in $\mathbb{C}$). Together with property 2.33(2) , this implies that F extends continuously to the closure $\overline{\mathcal{T}_F}$ of $\mathcal{T}_F$ in $\mathbb{C}$. We then denote

$$J(F) := \left\{ z \in \overline{\mathcal{T}_F} : F^j(z) \in \overline{\mathcal{T}_F} \text{ for all } j \geq 0 \right\}.$$

We say that a logarithmic transform F is of *disjoint type* if the boundaries of the tracts of F do not intersect the boundary of $\mathbb{H}_{\log L}$; i.e. if $\overline{\mathcal{T}_F} \subset \mathbb{H}_{\log L}$. In particular, in such case $J(F) = \bigcap_{n \geq 0} F^{-n}(\overline{\mathcal{T}_F})$. Recall that by Proposition 2.20, disjoint type functions in class $\mathcal{B}$ can be characterized as those for which there exists a disk $D \supset S(f)$ such that $\overline{f^{-1}(D)} \cap \partial D = \emptyset$. Hence, if F is a logarithmic transform of a disjoint type function $f \in \mathcal{B}$, then F is of disjoint type. Moreover, by the characterization mentioned, if f is of disjoint type, then $J(f) = \bigcap_{n \geq 0} f^{-n}(\mathcal{T}_f)$, see [RG16, Proposition 3.2], and so

$$\exp(J(F)) = J(f). \tag{2.5.2}$$

The equality (2.5.2) does not hold for all $f \in \mathcal{B}$. However, using a standard expansion estimate for logarithmic transforms derived from Koebe's $\frac{1}{4}$-theorem, see [EL92, Lemma 1], for $F\colon \mathcal{T}_F \to \mathbb{H}_{\log L}$ and all $z \in \mathcal{T}_F$,

$$|F'(z)| \geq \frac{1}{4\pi} \left(\operatorname{Re} F(z) - \log L\right). \tag{2.5.3}$$

Note that in particular, if $z \in \mathbb{H}_{\log L + 8\pi}$, (2.5.3) implies that $|F'(z)| \geq 2$. This allows us to see that a partial inclusion still holds in some cases:

[4]See footnote 2.

Observation 2.34. Let $F\colon \mathcal{T}_F \to \mathbb{H}_{\log L}$ be a logarithmic transform for $f \in \mathcal{B}$. If $X \subset J(F) \cap \mathbb{H}_{\log L+8\pi}$ and $F(X) \subset X$, then $\exp(X) \subset J(f)$.

Proof of observation: For the sake of contradiction, let us suppose that there exists $w \in \exp(X) \cap F(f)$. In particular, $w = \exp(z)$ for some $z \in X$, and by (2.5.3) together with the assumptions on X in the statement, $|(F^n)'(z)| \geq 2$ for all $n \geq 0$. By Theorem 2.17, $I(f) \subset J(f)$ and so $w \notin I(f)$. Since $\exp(F(z)) = f(\exp(z))$, it holds that

$$|f'(w)| = \frac{|F'(z)||f(w)|}{|w|} \geq 2\frac{|f(w)|}{|w|}.$$

Consequently, by this and using the chain rule, for any $n \geq 0$,

$$|(f^n)'(w)| \geq 2^n|f^n(w)|/|w| \geq 2^n L/|w|.$$

Thus, $|(f^n)'(w)| \to \infty$ as $n \to \infty$, which by Marty's theorem [Sch93, §3.3] contradicts the fact that $w \in F(f) \setminus I(f)$. $\square$

Thanks to Proposition 2.33(1) we are able to define, in an analogous way as we did in §2.4, symbolic dynamics for logarithmic transforms.

Definition 2.35 (External addresses for logarithmic transforms)**.** Let F be a logarithmic transform. An *(infinite) external address* is a sequence $\underline{s} = T_0T_1T_2\ldots$ of logarithmic tracts of F. We denote

$$J_{\underline{s}}(F) := \{z \in J(F) : F^n(z) \in \overline{T}_n \text{ for all } n \geq 0\}.$$

Moreover, $\operatorname{Addr}(J(F))$ is the set of external addresses $\underline{s}$ such that $J_{\underline{s}}(F) \neq \emptyset$.

Observation 2.36 (Correspondence between external addresses for F and f)**.** Let $f \in \mathcal{B}$ and let F be a logarithmic transform of f. In particular, the set of logarithmic tracts $\mathcal{T}_F$ of F is mapped under the exponential map to the set $\mathcal{T}_f$ of tracts of f. Let $\operatorname{Addr}(J(F))$ be defined with respect to the tracts in $\mathcal{T}_F$ and suppose that external addresses in the sense of Definition 2.21 have been defined for f with respect to some choice of fundamental domains. Then, each fundamental domain F_i lies in a tract $T(f)$ of f, and hence, if $T(F) \in \mathcal{T}_F$ is such that $\exp(T(F)) = T(f)$, then F_i has one preimage under the exponential map on each of the logarithmic tracts $\{T(F) + 2\pi i\mathbb{Z}\}$. Consequently, we can define the following equivalence relation on $\operatorname{Addr}(J(F))$: for a pair of addresses

$\underline{s} = T_0T_1T_2\dots$ and $\underline{\tau} = T_0'T_1'T_2'\dots$

$$\underline{s} \sim \underline{\tau} \qquad \Longleftrightarrow \qquad T_0' = T_0 + 2\pi ik \text{ for some } k \in \mathbb{Z} \text{ and } T_j = T_j' \text{ for all } j > 0, \tag{2.5.4}$$

and this leads to a 1-to-1 one correspondence between the set $\operatorname{Addr}(f)$ and $\operatorname{Addr}(J(F))/\!\sim$.

As a consequence of (2.5.3), the following expansion result holds for points with the same external address:

Proposition 2.37 (Expansion along orbits)**.** *Let $F : \mathcal{T} \to \mathbb{H}_{\log L}$ be a logarithmic transform of some $f \in \mathcal{B}$ such that $\overline{\mathcal{T}_F} \subset \mathbb{H}_{\log L+8\pi}$. For every $n \geq 0$ and $\underline{s} \in \operatorname{Addr}(J(F))$, if $z, w \in J_{\underline{s}}(F)$, then*

$$|z - w| \leq \frac{1}{2^n}|F^n(z) - F^n(w)|. \tag{2.5.5}$$

Proof. By the assumption $\overline{\mathcal{T}_F} \subset \mathbb{H}_{\log L+8\pi}$, using (2.5.3) it holds that $|F'(z)| \geq 2$ for all $z \in \mathcal{T}_F$. For any $T \in \mathcal{T}_F$, let $F_T^{-1} : \mathbb{H}_{\log L} \to T$ be the inverse branch of F onto the tract T. Then it holds that

$$|F_T^{-1}(v)| \leq \frac{1}{2} \quad \text{ for all } \quad v \in \mathbb{H}_{\log L}. \tag{2.5.6}$$

Moreover, the function F is of disjoint type by definition, and hence

$$J(F) = \bigcap_{n\geq 0} F^{-n}(\overline{\mathcal{T}_F}) \subset \mathcal{T}_F \subset \mathbb{H}_{\log L+8\pi}. \tag{2.5.7}$$

Let $\underline{s} = T_0T_1\dots$ and choose any pair of points $w, z \in J_{\underline{s}}(F)$. Let γ be the straight line joining $F(z)$ and $F(w)$. Since $\mathbb{H}_{\log L}$ is a right half plane, and hence a convex set, $\gamma \subset \mathbb{H}_{\log L}$. In particular, (2.5.6) holds for all points in γ, and moreover, since F_T^{-1} is a conformal isomorphism to its image, the curve $F_T^{-1}(\gamma)$ joins z and w. Consequently,

$$|z-w| \leq \ell_{eucl}(F_T^{-1}(\gamma)) = \int \left|(F_T^{-1})'(\gamma(t))\right||\gamma'(t)|dt \leq \frac{1}{2}\int |\gamma'(t)|dt = \frac{1}{2}|F(z)-F(w)|.$$

Since $z, w \in J_{\underline{s}}(F)$, for each $k \geq 0$ the points $F^k(z), F^k(w)$ belong to the same tract, and by (2.5.7), $F^{k+1}(z), F^{k+1}(w) \subset \mathbb{H}_{\log L}$. Hence, we can iteratively apply the same reasoning as before and (2.5.5) follows by induction. □

We conclude this section by providing the definition of a condition on logarithmic transforms that guarantees the existence of dynamic rays [RRRS11, Theorem

4.7]. We note that the Julia set of any disjoint type function for which a logarithmic transform satisfies this condition is a Cantor bouquet [BJR12, Corollary 6.3]. Compare to Proposition 4.39.

Definition 2.38 (Head-start condition [BJR12, Definition 4.1])**.** Let $F \in \mathcal{B}_{\log}$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a (not necessarily strictly) increasing continuous function with $\varphi(x) > x$ for all $x \in \mathbb{R}$. We say that F satisfies the *uniform head-start condition for* φ if:

1. For all tracts $T, T' \in \mathcal{T}_F$ and all $z, w \in \overline{T}$ with $F(z), F(w) \in \overline{T}'$,
$$\operatorname{Re} w > \varphi(\operatorname{Re} z) \implies \operatorname{Re} F(w) > \varphi(\operatorname{Re} F(z)) \,. \tag{2.5.8}$$
2. For all $\underline{s} \in \operatorname{Addr}(J(F))$ and for all distinct $z, w \in J_{\underline{s}}(F)$, there exists a constant $M \in \mathbb{N}$ such that either $\operatorname{Re} F^M(z) > \varphi(\operatorname{Re} F^M(w))$ or $\operatorname{Re} F^M(w) > \varphi(\operatorname{Re} F^M(z))$.

Chapter 3

Orbifold expansion and bounded Fatou components

3.1 Strongly postcritically separated functions

We start by defining and looking at the basic properties of the maps we study in this chapter.

Definition 3.1 (Postcritically separated, subhyperbolic and hyperbolic maps)**.** A transcendental entire function f is *postcritically separated* if $P_J := P(f) \cap J(f)$ is discrete and $P_F := P(f) \cap F(f)$ is compact. In the particular case when $P(f) \cap J(f)$ is finite, f is called *subhyperbolic*, and when $P(f) \cap J(f) = \emptyset$, f is *hyperbolic*.

Observation 3.2 (Dichotomy of points in P_J)**.** If f is postcritically separated, then any $p \in P_J$ is either (pre)periodic, or it escapes to infinity: indeed, if $p \notin I(f)$, then there exists a subsequence of points in the orbit of p that lies in a bounded set, and by discreteness of P_J on that set, the claim follows. In particular, if in addition $f \in \mathcal{B}$, there can be at most finitely many points in $S(f) \cap I(f)$.

Recall that for a holomorphic map $f : \widetilde{S} \to S$ between Riemann surfaces, the *local degree* of f at a point $z_0 \in \widetilde{S}$, denoted by $\deg(f, z_0)$, is the unique integer $n \geq 1$ such that the local power series development of f is of the form

$$f(z) = f(z_0) + a_n(z - z_0)^n + (\text{higher terms}),$$

where $a_n \neq 0$. Thus, $z_0 \in \mathbb{C}$ is a critical point of f if and only if $\deg(f, z_0) > 1$. We also say that f has *bounded criticality* in a set A if $\operatorname{AV}(f) \cap A = \emptyset$ and there exists a constant $M < \infty$ such that $\deg(f, z) < M$ for all $z \in A$.

Definition 3.3 (Strongly postcritically separated functions)**.** A postcritically separated transcendental entire map f is *strongly postcritically separated* with parameters (c, ϵ) if:

(a) f has bounded criticality in $J(f)$;

(b) for each $z \in J(f)$, $\#(\text{Orb}^+(z) \cap \text{Crit}(f)) \leq c$;

(c) for all distinct $z, w \in P_J$, $|z - w| \geq \epsilon \max\{|z|, |w|\}$.

Observation 3.4 (Separation of points in P_J)**.** In the definition of strongly postcritically separated map, (c) has the following implication: for every constant $K > 1$, there exists a constant $M > 0$, depending only on ϵ and K, so that

$$\#(P_J \cap \overline{A(r, Kr)}) \leq M \qquad \text{for all } r > 0. \tag{3.1.1}$$

To see this, note that the annulus $\overline{A(1, K)}$ admits at most some number M of points in P_J so that these points are at pairwise distance at least ϵ. Moreover, (c) implies that for any $r > 0$ and all distinct $w, z \in (P_J \cap A(r, Kr))$, $|z/r - w/r| > \epsilon$. The combination of these two facts justifies (3.1.1). In particular, since as r increases the annuli "$A(r, Kr)$" are of greater area, the orbit of any point $z \in S(f) \cap I(f)$ must converge to infinity at more than a *constant rate*, i.e., for all $C \in \mathbb{R}^+$, there must exist $n \geq 0$ so that $|f^{n+1}(z)| > |f^n(z)| + C$.

Remark. When f is subhyperbolic and Definition 3.3(a) holds, f is called *strongly subhyperbolic* [MB12, Definition 2.11]. Note that for subhyperbolic maps, conditions (b) and (c) in Definition 3.3 are trivially satisfied, and thus any strongly subhyperbolic map is a strongly postcritically separated one.

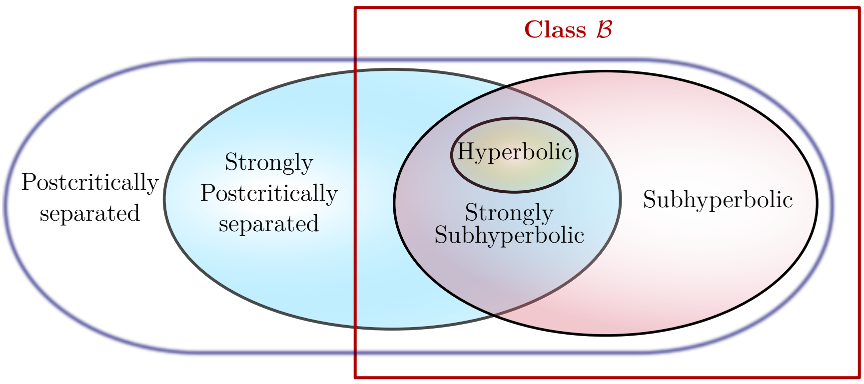

Figure 1: Illustration of the relationships between the classes of functions defined in this section.

Remark. If f is a strongly postcritically separated map, then so is f^n for all $n \geq 1$. This follows from the facts that $\text{AV}(f^n) = \bigcup_{i=0}^{n-1} f^i(\text{AV}(f))$, $\text{CV}(f^n) = \bigcup_{i=0}^{n-1} f^i(\text{CV}(f))$, $J(f^n) = J(f)$ and $P(f^n) = P(f)$.

Examples. The following functions belong to the classes of maps just defined:

- The exponential map is a postcritically separated map in class $\mathcal{B}$ that is neither strongly postcritically separated nor subhyperbolic, since its asymptotic value 0 escapes to infinity and is in its Julia set. See for example [SRG15].

- The function $f(z) := \pi \sinh(z)$ has only two critical values and no asymptotic values. Moreover, $P(f) = \{0, \pm\pi i\} \subset J(f)$. Thus, f is strongly subhyperbolic, and hence strongly postcritically separated. See [MB12, Appendix A] for a description of the dynamics of this map.

- For the function $f(z) := \cosh(z)$, $S(f) = \mathrm{CV}(f) = \{-1, 1\} \subset I(f)$. Moreover, $f \in \mathcal{B}$ and is strongly postcritically separated, but not subhyperbolic. See Section 5.1 for more details on the dynamics of this map. In particular, Theorems 1.1 and 4.64 apply to it.

- Let erf denote the error function [AS72, page 297] and let $\alpha \in \mathbb{C}$ be a complex solution to $\mathrm{erf}(\alpha) = 1$. In particular, we set $\alpha \approx 5.902 - 0.262i$. Let $g \colon \mathbb{C} \to \mathbb{C}$ given by

$$g(z) := \frac{2i\mathrm{Im}(\alpha)}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-w^2}\, dw + \mathrm{Re}(z) = i\mathrm{Im}(\alpha)\mathrm{erf}(z) + \mathrm{Re}(\alpha).$$

  Then $S(g) = \mathrm{AV}(g) = \{\alpha, \overline{\alpha}\}$, where $\overline{\alpha}$ is the complex conjugate of α. Since $\mathrm{erf}(\overline{\alpha}) = \overline{\mathrm{erf}(\alpha)} = \overline{\alpha}$, both asymptotic values are fixed points in $J(g)$. Hence, g is postcritically separated but not strongly postcritically separated. See [Six18a, page 7] for more details on functions constructed this way.

- The function $\cosh(z) - 1$ has as singular set two critical values, namely the point 0, which is superattracting, and the point -2, which belongs to the fast escaping set of the function. Hence, this is another example of a strongly postcritically separated function in class $\mathcal{B}$.

The types of Fatou components that might occur for postcritically separated functions follow from classical results included in §2.3:

Lemma 3.5 (Fatou components for postcritically separated maps)**.** *Let f be postcritically separated. Then $F(f)$ is either empty or might consist of a collection of attracting basins, Baker domains and escaping wandering domains. The number of attracting basins must be finite, and in the two latter cases, the domains do not contain singular values. In particular, P_F is contained in a finite union of attracting basins and every periodic cycle in $J(f)$ is repelling. If in addition $f \in \mathcal{B}$, $F(f)$ is either empty or a finite union of attracting basins.*

Remark. We do not claim the existence of examples of postcritically separated functions with wandering domains nor Baker domains. Instead, this lemma shows which types of Fatou components cannot occur for these functions.

Proof of Lemma 3.5. Compactness of P_F excludes parabolic components: suppose that $F(f)$ had a parabolic component U of period p, with a parabolic fixed point $z_0 \in \partial U$ such that for every $z \in U$, $f^{np}(z) \to z_0$ as $n \to \infty$. Then, by Proposition 2.16(1), there would exist $w \in S(f)$ that would also belong to some component in the cycle of U, and hence $f^j(w) \in U$ for some $0 \leq j < p$. But $\mathrm{Orb}^+(w) \subset P_F$, and simultaneously, $\mathrm{Orb}^+(w)$ would contain the subsequence $f^{p+j}(w), f^{2p+j}(w), \ldots$ converging to $z_0 \notin F(f)$, which would contradict compactness of P_F. Hence, $F(f)$ does not contain parabolic components. Note that by our assumptions of the discreteness of P_J and the compactness of P_F, $J(f) \cap P(f)' = \emptyset$. Thus, by Proposition 2.16(2), Siegel disks cannot occur for f.

If U is a wandering domain of f, since $J(f) \cap P(f)' = \emptyset$, by Proposition 2.16(3) it must be escaping. Since P_F is compact, $I(f) \cap P_F = \emptyset$, and so if Baker or escaping wandering domains occur for f, they cannot contain singular orbits. Hence, $P_F \subset \mathcal{A}(f)$, and by Proposition 2.11, P_F is contained in finitely many attracting basins. Since by Proposition 2.16(1) each cycle of attracting periodic components must contain a postsingular point, there cannot be any further attracting basins of $F(f)$. We have already discarded parabolic cycles in $J(f)$, as there are no parabolic components in $F(f)$. If z_0 was an irrationally indifferent periodic point in $J(f)$, then there would be a sequence $\{w_k\}_k \subset P(f)$ converging non-trivially to z_0. See [Mil06, Corollary 14.4]. Since P_J is discrete and P_F is contained in the union of finitely many attracting basins, this is impossible, and so all periodic cycles in $J(f)$ must be repelling. By [EL92] functions in class $\mathcal{B}$ do not have Baker domains or escaping wandering domains, and so for postcritically separated functions in this class only attracting basins can occur. □

3.2 Hyperbolic orbifold metrics

In the first part of this section we study the relation between the densities of the metrics of two hyperbolic orbifolds whenever one of them is holomorphically embedded in the other. More specifically, let $\widetilde{\mathcal{O}} := (\tilde{S}, \tilde{\nu})$ and $\mathcal{O} = (S, \nu)$ be hyperbolic orbifolds such that the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic. Note that in particular we are assuming that $\tilde{S} \subseteq S$. Then, recall that by Corollary 2.7 it holds that $\rho_{\widetilde{\mathcal{O}}}(z) \geq \rho_{\mathcal{O}}(z)$ for all unramified $z \in \widetilde{\mathcal{O}}$. The intuition behind this fact is the following: since a hyperbolic orbifold metric is defined as a pushforward of the hyperbolic metric in $\mathbb{D}$ with singularities at ramified points, its density tends to infinity both when approaching ramified points and when tending to the boundary of the underlying surface of the orbifold. Moreover, if w_0 is a ramified point, then the density function is of the form $|w - w_0|^{(1-m)/m}$ near it, where m is its ramified value. Note that for a fixed w_0, as m increases, the density function tends "faster" to infinity when we approach w_0. Hence, since $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ being holomorphic implies that $\tilde{S} \subseteq S$ and $\tilde{\nu}(z) \geq \nu(z)$ for all unramified $z \in \widetilde{\mathcal{O}}$, the desired inequality on their densities follows. This motivates the definition of the following set.

Definition 3.6 (Boundary of $\widetilde{\mathcal{O}}$ in $\mathcal{O}$)**.** Given a pair of orbifolds $\widetilde{\mathcal{O}} = (\tilde{S}, \tilde{\nu})$ and $\mathcal{O} = (S, \nu)$ such that the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, we define the *boundary of* $\widetilde{\mathcal{O}}$ *in* $\mathcal{O}$ as the set

$$\mathbb{B}_{\widetilde{\mathcal{O}}}^{\mathcal{O}} := \partial \tilde{S} \cup \left\{ z \in \tilde{S} : \tilde{\nu}(z) > \nu(z) \right\}.$$

Remark. If $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, then $\mathbb{B}_{\widetilde{\mathcal{O}}}^{\mathcal{O}} \neq \emptyset$ if and only if $\tilde{S} \subsetneq S$, or $S = \tilde{S}$ and the inclusion is not an orbifold covering map. Moreover, see Observation 2.8, the quotient $\rho_{\widetilde{\mathcal{O}}}(z)/\rho_{\mathcal{O}}(z)$ is well-defined for all unramified $z \in \widetilde{\mathcal{O}}$.

Under the conditions of Definition 3.6, Theorem 1.12 provides bounds for the quotient of densities in terms of the $\mathcal{O}$-distance between a point $z \in \tilde{S}$ and the set $\mathbb{B}_{\widetilde{\mathcal{O}}}^{\mathcal{O}}$. This is inspired in [MBRG13, Proposition 3.4], where an analogous result is shown to hold for hyperbolic Riemann surfaces. Let us restate Theorem 1.12 in a more precise version:

Theorem 1.12 (Relative densities of hyperbolic orbifolds)**.** *Let* $\widetilde{\mathcal{O}} := (\tilde{S}, \tilde{\nu})$ *and* $\mathcal{O} := (S, \nu)$ *be hyperbolic orbifolds such that the inclusion* $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ *is holomorphic. Let* $z \in \widetilde{\mathcal{O}}$ *be unramified and suppose that* $R := d_{\mathcal{O}}(z, \mathbb{B}_{\widetilde{\mathcal{O}}}^{\mathcal{O}}) < \infty$*. Then,*

$$1 < \frac{e^R}{\sqrt{e^{2R} - 1}} \leq \frac{\rho_{\widetilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} \leq 1 + \frac{2}{e^R - 1}. \tag{3.2.1}$$

Remark. The exact dependence of the bounds on R is not relevant for our purposes, but instead, we are interested in the fact that the quotient of densities depends only on R and is bounded away from 1. See Figure 2. Still, we point out that the proof will show that the bounds are sharp, in the sense that they can be attained.

Proof of Theorem 1.12. We can assume without loss of generality that the surfaces $\tilde{S}$ and S are both connected, since otherwise the same argument applies

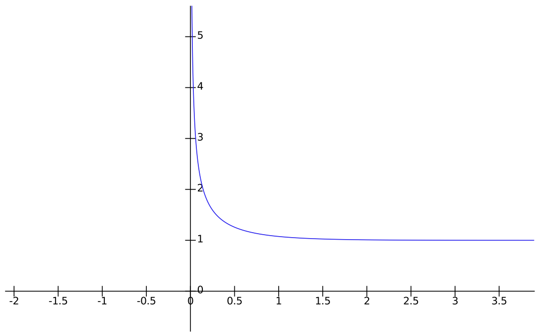

Figure 2: Plot of the function $\Lambda(R) := \mathrm{e}^R/\sqrt{e^{2R}-1}$, that provides a lower bound for the quotient of densities in the setting of Theorem 1.12. Observe that $\Lambda(R) > 1$ for all $R > 0$.

component-wise. For the point z fixed in the statement of the proposition, let $\pi\colon \mathbb{D} \to \mathcal{O}$ be a covering map with $\pi(0) = z$. In particular, by definition of orbifold covering map, for any $x \in \mathbb{D}$,

$$\nu(\pi(x)) = \deg(\pi, x) \cdot \nu_{\mathbb{D}}(x) = \deg(\pi, x), \tag{3.2.2}$$

as $\nu_{\mathbb{D}} \equiv 1$ by definition. Since by assumption $z \in \tilde{S} \subseteq S$, there exists a connected component of $\pi^{-1}(\tilde{S})$ that contains the point 0. We shall denote this component by $\widehat{D}$. Since $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, $\nu(w)$ divides $\tilde{\nu}(w)$ for all $w \in \widetilde{\mathcal{O}}$, and so, using (3.2.2), we can define a ramification map $\hat{\nu}\colon \widehat{D} \to \mathbb{N}_{\geq 1}$ as

$$\hat{\nu}(x) := \frac{\tilde{\nu}(\pi(x))}{\deg(\pi, x)}. \tag{3.2.3}$$

Note that by (3.2.2) and since $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, for each $x \in \widehat{D}$, $\hat{\nu}(x) > 1$ if and only if $\pi(x) \in \left\{w \in \widetilde{\mathcal{O}} : \tilde{\nu}(w) > \nu(w)\right\}$. Since $\left\{w \in \widetilde{\mathcal{O}} : \tilde{\nu}(w) > \nu(w)\right\}$ is a discrete set, as $\tilde{\nu}$ is the ramification map of an orbifold, $\hat{\mathcal{O}} := (\widehat{D}, \hat{\nu})$ is also a Riemann orbifold. Observe that by definition, the restriction $\pi|_{\widehat{D}}\colon \hat{\mathcal{O}} \to \tilde{\mathcal{O}}$ is an orbifold covering map.

Since by assumption $d_{\mathcal{O}}(z, \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}) = R$, by definition of the set $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$, there must exist at least one point $z_2 \in \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$ such that $d_{\mathcal{O}}(z, z_2) = R$. In particular, $z_2 \in S$. Let us connect z_2 to z by a geodesic (in the metric of $\mathcal{O}$) of length R. By lifting this geodesic to the unit disc using the map π, we see using Theorem 2.6 that there exists $w \in \mathrm{cl}(\widehat{D})$ such that $\mathrm{dist}_{\mathbb{D}}(0, w) = R$. By pre-composing with a rotation, we can assume that w is a positive real number. We recall that the densities of the hyperbolic metric on $\mathbb{D}_r$ for some $r \in \mathbb{R}^+$ and $\mathbb{D}^*$, are respectively given by

$$\rho_{\mathbb{D}_r}(x) = \frac{2}{r\,(1-|x|^2/r^2)} \quad \text{and} \quad \rho_{\mathbb{D}^*}(x) = \frac{1}{|x|\cdot|\log|x||}. \tag{3.2.4}$$

Since π is a covering map, by Theorem 2.6, $d_{\mathbb{D}}(x,y) \geq d_{\mathcal{O}}(\pi(x),\pi(y))$ for all $x, y \in \mathbb{D}$ and hence, by the choice of w, the disc (in the hyperbolic metric on $\mathbb{D}$) of radius R centred at the origin is contained in $\widehat{D}$, and particular is an Euclidean disc of radius w. Moreover, by definition of the constant R, $\hat{\nu}(z) = 1$ for all $z \in \mathbb{D}_w \subset \widehat{D}$, and thus, if we regard $\mathbb{D}_w$ as a hyperbolic orbifold with ramification map constant and equal to 1, the inclusion $\mathbb{D}_w \hookrightarrow \hat{\mathcal{O}}$ is holomorphic. In particular, by Corollary 2.7 $\rho_{\hat{\mathcal{O}}}(x) \leq \rho_{\mathbb{D}_w}(x)$ for all $x \in \mathbb{D}_w$. Thus, using Theorem 2.6, (3.2.4) and recalling that $\pi(0) = z$,

$$\frac{\rho_{\tilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} = \frac{|\pi'(0)|\cdot\rho_{\tilde{\mathcal{O}}}(\pi(0))}{|\pi'(0)|\cdot\rho_{\mathcal{O}}(\pi(0))} = \frac{\rho_{\hat{\mathcal{O}}}(0)}{\rho_{\mathbb{D}}(0)} \leq \frac{\rho_{\mathbb{D}_w}(0)}{\rho_{\mathbb{D}}(0)} = \frac{1}{w}. \tag{3.2.5}$$

We have obtained an upper bound for the relative densities at z in terms of the value w. In order to get a lower bound, we divide the proof into two cases depending on whether $\hat{\nu}(w) = 1$ or $\hat{\nu}(w) > 1$. In the first case, $z_2 = \pi(w) \in \partial\tilde{S}$, and so $w \in \partial\widehat{D}$. In particular, $\widehat{D} \subset \mathbb{D} \setminus \{w\}$, and so the inclusion $\hat{\mathcal{O}} \hookrightarrow (\mathbb{D}\setminus\{w\}, \rho_{\mathbb{D}\setminus\{w\}})$ is holomorphic, where $\rho_{\mathbb{D}\setminus\{w\}}$ is the constant function equal to 1. Therefore, by Corollary 2.7, $\rho_{\hat{\mathcal{O}}}(x) \geq \rho_{\mathbb{D}\setminus\{w\}}(x)$ for all unramified $x \in \hat{\mathcal{O}}$. Consider the Möbius transformation $T\colon \mathbb{D} \to \mathbb{D}$ given by $T(x) := \frac{x-w}{wx-1}$, which in particular satisfies $T(w) = 0$ and $T(0) = w$. The restriction $T|_{\mathbb{D}\setminus\{w\}}$ is a covering map for the orbifold with underlying surface $\mathbb{D}^*$ and ramification map constant equal to one. Then, using Theorem 2.6, (3.2.4) and (3.2.5),

$$\frac{\rho_{\tilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} = \frac{\rho_{\hat{\mathcal{O}}}(0)}{\rho_{\mathbb{D}}(0)} \geq \frac{\rho_{\mathbb{D}\setminus\{w\}}(0)}{\rho_{\mathbb{D}}(0)} = \frac{|T'(0)|\cdot\rho_{\mathbb{D}^*}(T(0))}{|T'(0)|\cdot\rho_{\mathbb{D}}(T(0))} = \frac{\rho_{\mathbb{D}^*}(w)}{\rho_{\mathbb{D}}(w)} = \frac{1-w^2}{2w|\log w|}. \tag{3.2.6}$$

For the second case, that is, whenever $k := \hat{\nu}(w) \geq 2$, we define the orbifold $\mathcal{O}_w^k := (\mathbb{D}, \mu)$ with $\mu(w) = k$ and $\mu \equiv 1$ elsewhere. Then, the inclusion $\hat{\mathcal{O}} \hookrightarrow \mathcal{O}_w^k$ is holomorphic, and so by Corollary 2.7, $\rho_{\hat{\mathcal{O}}}(x) \geq \rho_{\mathcal{O}_w^k}(x)$ for all $x \in \hat{D}$. Thus, using (3.2.5),

$$\frac{\rho_{\tilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} = \frac{\rho_{\hat{\mathcal{O}}}(0)}{\rho_{\mathbb{D}}(0)} \geq \frac{\rho_{\mathcal{O}_w^k}(0)}{\rho_{\mathbb{D}}(0)}. \tag{3.2.7}$$

Let $\mathcal{O}_0^k$ be the orbifold with underlying surface the unit disc and signature (k), being 0 its only ramified point. Let $f\colon \mathbb{D} \to \mathcal{O}_0^k$ be the covering map given by $f(x) = x^k$. Then, $T \circ f\colon \mathbb{D} \to \mathcal{O}_w^k$ is an orbifold covering map, and thus by

Theorem 2.6,

$$|f'(x)|\rho_{\mathcal{O}_0^k}(f(x)) = \rho_{\mathbb{D}}(x) = \left|T^{'}(f(x))\right| \cdot |f'(x)|\,\rho_{\mathcal{O}_w^k}(T(f(x))).$$

Hence, if we choose any $x \in \mathbb{D}$ such that $f(x) = w$, using that $T(w) = 0$, we get that $\rho_{\mathcal{O}_0^k}(w) = |T'(w)|\rho_{\mathcal{O}_w^k}(0)$. Arguing similarly, $\rho_{\mathbb{D}}(w) = |T'(w)|\rho_{\mathbb{D}}(0)$. Thus, substituting in (3.2.7),

$$\frac{\rho_{\tilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} \geq \frac{\rho_{\mathcal{O}_w^k}(0)}{\rho_{\mathbb{D}}(0)} = \frac{\rho_{\mathcal{O}_0^k}(w)}{\rho_{\mathbb{D}}(w)}. \tag{3.2.8}$$

We aim to get a lower bound for $\rho_{\mathcal{O}_0^k}(w)/\rho_{\mathbb{D}}(w)$ independent of the value k. We can compute the density of the induced metric in $\mathcal{O}_0^k$ using that $\rho_{\mathbb{D}}(x) = |f'(x)|\rho_{\mathcal{O}_0^k}(f(x))$ and (3.2.4). Since $f(x) = x^k = u$ implies $x^{k-1} = u^{\frac{k-1}{k}}$, we get that for each $u \in \mathbb{D}$,

$$\rho_{\mathcal{O}_0^k}(u) = \frac{2}{k|u|^{\frac{k-1}{k}}(1-|u|^{\frac{2}{k}})}. \tag{3.2.9}$$

Thus, if we make the change of variables $q \coloneqq 1/k$, $r \coloneqq 1/w$, we are aiming to find a lower bound independent of q for

$$\frac{\rho_{\mathcal{O}_0^k}(w)}{\rho_{\mathbb{D}}(w)} = \frac{1-w^2}{kw^{\frac{k-1}{k}}(1-w^{\frac{2}{k}})} = \frac{q(1-r^{-2})r}{r^q - r^{-q}}, \quad \text{where} \quad q \in (0, 1/2] \quad \text{and} \quad r > 1. \tag{3.2.10}$$

Observe that for each fixed value of r, the last quotient above is strictly decreasing in q. This can be seen by considering for each $r > 1$ the functions $f_r \colon (0, 1/2] \to \mathbb{R}$ given by

$$f_r(q) \coloneqq \frac{q}{r^q - r^{-q}} = \frac{q}{\sinh(q \log r)} = \frac{s}{\log r \sinh(s)},$$

where we have made the change of variables $s = q \log r$. Let $h(s) \coloneqq s/\sinh(s)$ and note that $h'(s) = (\sinh(s) - s\cosh(s))/\sinh^2(s)$ is always negative as $\tanh(s) < s$ when s is positive. Thus, the same holds for $f_r'(q)$ and so each function f_r is strictly decreasing in q. Substituting in (3.2.10),

$$\frac{\rho_{\mathcal{O}_0^k}(w)}{\rho_{\mathbb{D}}(w)} \geq \frac{(1-r^{-2})r}{2(r^{1/2} - r^{-1/2})} = \frac{r^{-1/2}(1+r^{-1})}{2} = \frac{1+w}{2\sqrt{w}} \quad \text{for each} \quad w < 1. \tag{3.2.11}$$

Thus, putting together equations (3.2.6), (3.2.8) and (3.2.11) we get that for the point z in the statement,

$$\frac{\rho_{\tilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} \geq \min\left\{\frac{1-w^2}{2w|\log w|}, \frac{1+w}{2\sqrt{w}}\right\} = \frac{1+w}{2\sqrt{w}} > 1. \tag{3.2.12}$$

Finally, (3.2.1) is obtained recalling that the hyperbolic distance between 0 and any point $z \in \mathbb{D}$ is given by $\log \frac{1+|z|}{1-|z|}$. In our case, $d_{\mathbb{D}}(0, w) = \log \frac{1+w}{1-w} = R$, and so $w = \frac{\mathrm{e}^R - 1}{\mathrm{e}^R + 1}$. Substituting accordingly in equations (3.2.5) and (3.2.12) the desired bounds are achieved. □

The second goal of this section is to prove Theorem 1.13. In order to achieve this, we will first prove in Theorem 3.9, for orbifolds with the same number of ramified points, all with the same ramification value, that if these ramified points are "continuously perturbed", the orbifold metric of the new orbifold is a "continuous perturbation" of the metric of the original one. It is possible that these results have appeared before in the literature of orbifolds, but since a reference has not been located, we present proofs that use quasiconformal maps. We refer to [LV73, Vuo88] for definitions.

We start by fixing the type of orbifolds that we shall consider. Namely, those for which their ramified points are at least at a certain (given) Euclidean distance from each other.

Definition 3.7 (Orbifolds associated to vectors)**.** Given a compact subset A of a Jordan domain $U \subsetneq \mathbb{C}$ and constants $N, \tilde{N} \in \mathbb{N}_{\geq 1}$ and $r > 0$, denote

$$\mathcal{T}_r^N(A) := \left\{(w_1, \ldots, w_N) \in A^N : |w_i - w_j| \geq r \text{ for all } i \neq j\right\}. \tag{3.2.13}$$

Each $\boldsymbol{w} = (w_1, \ldots, w_N) \in \mathcal{T}_r^N(A)$ has an *associated orbifold* $\mathcal{O}_{\boldsymbol{w}}^{\tilde{N}} := (U, \nu_{\boldsymbol{w}})$, with

$$\nu_{\boldsymbol{w}}(z) := \begin{cases} \tilde{N} & \text{if } z = w_i \text{ for some } 1 \leq i \leq N, \\ 1 & \text{otherwise.} \end{cases} \tag{3.2.14}$$

Remark. By Theorem 2.4, for any orbifold $\mathcal{O}_{\boldsymbol{w}}^{\tilde{N}}$ as in Definition 3.7, since $U \subsetneq \mathbb{C}$, its universal cover is $\mathbb{D}$, and so the distance function $d_{\mathcal{O}_{\boldsymbol{w}}^{\tilde{N}}}$ is well-defined.

Observation 3.8 ($\mathcal{T}_r^N(A)$ is compact)**.** Under the conditions of Definition 3.7, the set $\mathcal{T}_r^N(A)$ is a closed subset of the compact set A^N, and hence it is compact. To see this, let $\boldsymbol{w} = (w_1, \ldots, w_N)$ be a limit point of $\mathcal{T}_r^N(A)$ and suppose that $\boldsymbol{w} \notin \mathcal{T}_r^N(A)$. Then, $|w_i - w_j| = d < r$ for some $i \neq j$. Choose ϵ, $0 < \epsilon < r - d$. Since $\boldsymbol{w}$ is a limit point, there exists $\boldsymbol{v} = (v_1, \ldots, v_N) \in \mathcal{T}_r^N(A)$ such that $v_i \in \mathbb{D}_{\epsilon/2}(w_i)$ and $v_j \in \mathbb{D}_{\epsilon/2}(w_j)$. But then $|v_i - v_j| \leq |v_i - w_i| + |w_i - w_j| + |w_j - v_j| < d + \epsilon < r$, which contradicts $\boldsymbol{v} \in \mathcal{T}_r^N(A)$.

In the following theorem, we see that continuous perturbations of a vector $\boldsymbol{w} \in \mathcal{T}_r^N(A)$ lead to continuous perturbations on the distance function $d_{\mathcal{O}_{\boldsymbol{w}}}$ of its associated orbifold. Compare to [MB12, Theorem 4.2] for a similar argument when a single ramified point of an orbifold is perturbed.

Theorem 3.9 (Continuity of orbifold metrics under perturbations)**.** *Let A be a compact subset of a Jordan domain U. Let $N, \tilde{N} \in \mathbb{N}_{\geq 1}$ and $r > 0$. Then, the function $h\colon A^2 \times \mathcal{T}_r^N(A) \to \mathbb{R}$ given by*

$$h(p, q, w_1, \dots, w_N) := d_{\mathcal{O}_{\boldsymbol{w}}}(p, q)$$

is continuous, where $\boldsymbol{w} := (w_1, \dots, w_N) \in \mathcal{T}_r^N(A)$ and $\mathcal{O}_{\boldsymbol{w}} := \mathcal{O}_{\boldsymbol{w}}^{\tilde{N}}$ is its associated orbifold.

Proof. Since the domain of the function h is a metric space, the notions of continuity and sequential continuity for h are equivalent. Thus, we will prove the theorem by showing that for a fixed but arbitrary $\boldsymbol{x} := (p, q, w_1, \dots, w_N) \in A^2 \times \mathcal{T}_r^N(A)$, if $\{\boldsymbol{x_k} := (p^k, q^k, w_1^k, \dots, w_N^k)\}_{k\geq 1}$ is a sequence of points in $A^2 \times \mathcal{T}_r^N(A)$ such that $\boldsymbol{x_k} \to \boldsymbol{x}$ as $k \to \infty$, then $h(\boldsymbol{x_k}) \to h(\boldsymbol{x})$. That is, if $\boldsymbol{w} := (w_1, \dots, w_N)$ and for each $k \geq 1$, $\boldsymbol{w_k} := (w_1^k, \dots, w_N^k)$, then we will prove continuity of h by showing that

$$d_{\mathcal{O}_{\boldsymbol{w_k}}}(p^k, q^k) \to d_{\mathcal{O}_{\boldsymbol{w}}}(p, q) \quad \text{as} \quad k \to \infty. \tag{3.2.15}$$

By conformal conjugacy, we may assume without loss of generality that $0 \in A$ and $w_j \neq 0$ for all $1 \leq j \leq N$. Then, since $\boldsymbol{w} \in \mathcal{T}_r^N(A)$, we can choose $\epsilon < r/2$ so that all disks in the set

$$\{\mathbb{D}_\epsilon\} \cup \{\mathbb{D}_\epsilon(w_j) : 1 \leq j \leq N\}$$

are pairwise disjoint and contained in A. Since by assumption $\boldsymbol{x_k} \to \boldsymbol{x}$ as $k \to \infty$, there exists $K > 0$ such that $w_j^k \subset \mathbb{D}_\epsilon(w_j)$ for all $k \geq K$ and $1 \leq j \leq N$. Moreover, for each $k \geq K$ and $1 \leq j \leq N$, we define a quasiconformal map $\varphi_j^k\colon \mathbb{D}_\epsilon(w_j) \to \mathbb{D}_\epsilon(w_j)$ that satisfies $\varphi_j^k(w_j^k) = w_j$. With that aim, let $H_j^k\colon \mathbb{D}_\epsilon(w_j) \to \mathbb{H}$ be the unique Riemann map such that $H_j^k(w_j) = i$ and so that $H_j^k(w_j^k)$ lies in the positive imaginary axis. Recall that $\mathbb{H}$ denotes the upper half-plane. In particular, $H_j^k(w_j^k) = h_j^k i$, where $h_j^k := \mathrm{e}^{d_j(w_j^k, w_j)}$ and d_j denotes the hyperbolic metric in $\mathbb{D}_\epsilon(w_j)$. Define $L_j^k : \mathbb{H} \to \mathbb{H}$ as $L_j^k(z) := \mathrm{Re}(z) + h_j^k \mathrm{Im}(z) i$. Note that L_j^k is a h_j^k-quasiconformal self-map of $\mathbb{H}$. We then define

$$\varphi_j^k\colon \mathbb{D}_\epsilon(w_j) \to \mathbb{D}_\epsilon(w_j) \quad \text{as} \quad \varphi_j^k := \left(H_j^k\right)^{-1} \circ L_j^k \circ H_j^k.$$

If follows from the definition of the functions involved, that φ_j^k extends continuously to $\partial\mathbb{D}_\epsilon(w_j)$ as the identity map. Hence, the map $\varphi_k\colon U \to U$ given by

$$\varphi_k(z) := \begin{cases} \varphi_j^k(z) & \text{if } z \in \mathbb{D}_\epsilon(w_j) \text{ for some } 1 \leq j \leq N, \\ z & \text{otherwise,} \end{cases} \tag{3.2.16}$$

is well-defined and continuous. In fact, φ_k is a $K(k)$-quasiconformal map, where $K(k) := \max_{i=1}^N\{h_j^k\}$ (see for example [GŚ98]), and moreover,

$$\varphi_k|_{\mathbb{D}_\epsilon} \equiv \mathrm{id}|_{\mathbb{D}_\epsilon} \qquad \text{for all } k \geq K. \tag{3.2.17}$$

In particular, $K(k) \to 1$ and $\varphi_k \to \mathrm{id}$ as k tends to infinity. Let $\pi : \mathbb{D} \to \mathcal{O}_{\boldsymbol{w}}$ and $\pi_k : \mathbb{D} \to \mathcal{O}_{\boldsymbol{w_k}}$ be orbifold covering maps, normalized such that it holds $\pi(0) = \pi_k(0) = 0$ and $\arg(\pi'(0)) = \arg(\pi_k'(0))$. Note that as an orbifold map, $\varphi_k\colon \mathcal{O}_{\boldsymbol{w_k}} \to \mathcal{O}_{\boldsymbol{w}}$ is a homeomorphism that preserves ramified points, that is,

$$\nu_{\boldsymbol{w}}(z) > 1 \iff \nu_{\boldsymbol{w_k}}(\varphi_k(z)) > 1. \tag{3.2.18}$$

Thus, we can *lift* φ_k to a homeomorphism $\Phi_k\colon \mathbb{D} \to \mathbb{D}$ such that

$$\pi \circ \Phi_k = \varphi_k \circ \pi_k \qquad \text{and} \qquad \Phi_k(0) = 0. \tag{3.2.19}$$

Note that Φ_k is also locally $K(k)$-quasiconformal, since both π and π_k are holomorphic and φ_k is a $K(k)$-quasiconformal map. In addition, by combining our assumptions on the derivatives of the maps involved at 0, (3.2.17), (3.2.19) and that $\varphi_k \to \mathrm{id}$, we have that $|\Phi_k'(0)| \to 1$ as $k \to \infty$. Since the space of quasiconformal self-maps of the disk fixing zero and satisfying the last property is compact, the sequence $\{\Phi_k\}_k$ converges locally uniformly to the identity as $k \to \infty$. Hence, by (3.2.18) and (3.2.19), π_k converges locally uniformly to π as $k \to \infty$.

Recall that our goal is to prove (3.2.15). Note that since $\{\pi_k\}_k$ and π are orbifold covering maps, by Theorem 2.6 they are local isometries. Hence, instead of proving (3.2.15) using the orbifold metrics in $\mathcal{O}_{\boldsymbol{w_k}}$ and $\mathcal{O}_{\boldsymbol{w}}$, we will prove an analogue of (3.2.15) for preimages of the points p, q, p^k, q^k under the covering maps $\{\pi_k\}_k$ and π. More precisely, let us choose δ small enough so that there exist respective connected components V_p and V_q of $\pi^{-1}(\mathbb{D}_\delta(p))$ and $\pi^{-1}(\mathbb{D}_\delta(q))$ containing a single preimage of p and of q respectively. That is, $\pi^{-1}(p) \cap V_p =: \{\tilde{p}\}$ and $\pi^{-1}(q) \cap V_q =: \{\tilde{q}\}$. In addition, for each $k \geq K$ let us consider the holomorphic functions $\pi_k^p : V_p \to U$ and $\pi_k^q : V_q \to U$ given by $\pi_k^p(z) := \pi_k(z) - p$ and $\pi_k^q(z) := \pi_k(z) - q$. Then, the sequences $\{\pi_k^p\}_k$ and $\{\pi_k^q\}_k$ converge uniformly in compact subsets to the functions $\pi|_{V_p} - p$ and $\pi|_{V_p} - q$, which have respectively unique zeros at p and q. Then, by Hurwitz's theorem, for each k large enough, there exist points $\{\tilde{p}_{\pi_k}\} := \pi_k^{-1}(p) \cap V_p \cap \mathbb{D}_{\delta_1}(\tilde{p})$ and $\{\tilde{q}_{\pi_k}\} := \pi_k^{-1}(q) \cap V_q \cap \mathbb{D}_{\delta_1}(\tilde{q})$ for some δ_1 small enough. In particular,

$$\tilde{p}_{\pi_k} \xrightarrow{k\to\infty} \tilde{p} \quad \text{and} \quad \tilde{q}_{\pi_k} \xrightarrow{k\to\infty} \tilde{q}. \tag{3.2.20}$$

Note that for each $k \geq K$, $\tilde{p}_{\pi_k}$ is a preimage of p under π_k, rather than a preimage of p^k under π_k, and hence the proof is not concluded just yet. However, since by assumption $p^k \xrightarrow{k\to\infty} p$ and $q^k \xrightarrow{k\to\infty} q$, for every k sufficiently large, $\pi_k^{-1}(p^k) \cap V_p =: \{P_k\}$, $\pi_k^{-1}(q^k) \cap V_q =: \{Q_k\}$, and in addition

$$|P_k - \tilde{p}_{\pi_k}| \xrightarrow{k\to\infty} 0 \quad \text{and} \quad |Q_k - \tilde{q}_{\pi_k}| \xrightarrow{k\to\infty} 0. \tag{3.2.21}$$

Thus, as a combination of (3.2.20) and (3.2.21), and using that for compact subsets of $\mathbb{D}$, the Euclidean and hyperbolic metrics are equivalent, we have that

$$d_{\mathbb{D}}(P_k, Q_k) \to d_{\mathbb{D}}(\tilde{p}, \tilde{q}) \quad \text{as} \quad k \to \infty,$$

which is equivalent to (3.2.15), as we wanted to show. $\square$

Theorem 1.13 now becomes a consequence of the preceding one together with Theorem 1.12. We restate it here for ease of exposition.

Theorem 1.13 (Distances are uniformly bounded across certain orbifolds)**.** *Given a compact subset A of a Jordan domain U and constants $r > 0$ and $c, M \in \mathbb{N}_{\geq 1}$, there exists a constant $R := R(U, A, r, c, M) > 0$ such that for every orbifold $\mathcal{O}$ with underlying surface U and at most M ramified points, each with ramification value smaller than or equal to c, and such that the Euclidean distance between any two of them is at least r, it holds that*

$$d_{\mathcal{O}}(p, q) < R \quad \textit{for every} \quad p, q \in A.$$

Proof. For each $N = 1, \ldots, M$, we apply Theorem 3.9 to the compact set A, the domain $U \supset A$ and constants N and $\tilde{N} = c!$. Then, Theorem 3.9 asserts that for each N, the function $h\colon A^2 \times \mathcal{T}_r^N(A) \to \mathbb{R}$ of its statement is continuous and defined on a compact set. Hence, for each N, there exists a constant R_N such that for any orbifold $\mathcal{O}_{\boldsymbol{w}} = (U, \nu_{\mathcal{O}_{\boldsymbol{w}}})$ with $\boldsymbol{w} \in \mathcal{T}_r^N(A)$ and $\nu_{\mathcal{O}_{\boldsymbol{w}}}$ as specified in (3.2.14), $d_{\mathcal{O}_{\boldsymbol{w}}}(p, q) < R_N$ for all $p, q \in A$. Note that by Corollary 2.7, the same bound holds for any orbifold with N ramified points in A with ramification degrees between 1 and c. This is because the inclusion map would be holomorphic as their ramification values divide $c!$, and the same argument applies if the orbifold has no ramified points in A. Let us define $\tilde{R} := \max_{N \leq M} R_N$. Then, if $\hat{\mathcal{O}} := (U, \hat{\nu})$ is any orbifold with at most M ramified points, any two at Euclidean distance at least r, all lying in A and each of them with ramification value at most c, then

$$d_{\hat{\mathcal{O}}}(p,q) < \tilde{R} \quad \text{for all} \quad p,q \in A. \tag{3.2.22}$$

Let us fix any orbifold $\mathcal{O} := (U, \nu)$ satisfying the hypotheses of the statement of this theorem. Moreover, let us fix $\hat{\mathcal{O}} = (U, \hat{\nu})$ with $\hat{\nu} \equiv \nu|_A$ in A and $\hat{\nu} \equiv 1$ in $U \setminus A$, and note that (3.2.22) holds for $\hat{\mathcal{O}}$.

Let $W := \{z \in U : d_{\hat{\mathcal{O}}}(A, z) < \tilde{R}\}$ and define the orbifold $\widetilde{\mathcal{O}} := (W, \nu|_W)$, with $\nu|_W$ being the restriction of ν to W. Observe that $A \Subset W$ and that the inclusions $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ and $\widetilde{\mathcal{O}} \hookrightarrow \hat{\mathcal{O}}$ are holomorphic. In particular, the boundary of $\widetilde{\mathcal{O}}$ and $\hat{\mathcal{O}}$, denoted $\mathbb{B}^{\hat{\mathcal{O}}}_{\widetilde{\mathcal{O}}}$, consists of all ramified points of $\widetilde{\mathcal{O}}$ lying in $W \setminus A$ together with ∂W. Then, by definition of W, for all $z \in A$, $d_{\hat{\mathcal{O}}}(z, \mathbb{B}^{\hat{\mathcal{O}}}_{\widetilde{\mathcal{O}}}) < \tilde{R}$, and by Theorem 1.12, for all unramified $z \in A$, $\frac{\rho_{\widetilde{\mathcal{O}}}(z)}{\rho_{\hat{\mathcal{O}}}(z)} \geq 1 + \frac{2}{e^{\tilde{R}} - 1} =: K$. Moreover, if γ is a geodesic in the metric of $\hat{\mathcal{O}}$ joining two points $p, q \in A$, again by the choice of W, γ must be totally contained in W, and hence in $\widetilde{\mathcal{O}}$. Thus,

$$d_{\hat{\mathcal{O}}}(p,q) = \int |\gamma'(t)| \rho_{\hat{\mathcal{O}}}(\gamma(t)) dt \geq \frac{1}{K} \int |\gamma'(t)| \rho_{\widetilde{\mathcal{O}}}(\gamma(t)) dt \geq \frac{1}{K} d_{\widetilde{\mathcal{O}}}(p,q).$$

By this and by Corollary 2.7, for all $p, q \in A$

$$d_{\mathcal{O}}(p,q) \leq d_{\widetilde{\mathcal{O}}}(p,q) \leq K d_{\hat{\mathcal{O}}}(p,q) \leq K \cdot \tilde{R} =: R.$$

Since the constant K does not depend on the domain W but only on $\tilde{R}$, the statement follows. ☐

3.3 Uniform expansion

This section is devoted to the proof of Theorem 1.1: for each strongly postcritically separated function $f \in \mathcal{B}$, we define a pair of hyperbolic orbifolds $(\widetilde{\mathcal{O}}, \mathcal{O})$ so that in particular their underlying surfaces contain $J(f)$ and so that $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map. In order to construct these orbifolds, we take into account Corollary 2.9. That is, a first step towards *expansion* requires, in addition to the conditions above, that the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic. Then, the combination of the conditions on the inclusion being holomorphic and f being a covering map, (i.e. merging formulae (2.2.1) and (2.2.2)) implies that the ramification map ν of $\mathcal{O}$ must satisfy

$$\deg(f, z) \cdot \nu(z) \text{ divides } \nu(f(z)) \text{ for all } z \in \widetilde{\mathcal{O}}. \tag{3.3.1}$$

In other words, if $z \in \widetilde{\mathcal{O}}$, then $\deg(f, p) \cdot \nu(p)$ divides $\nu(z)$ for all $p \in f^{-1}(z)$.

Remark. Note that if $J(f)$ is in the underlying surfaces of $\widetilde{\mathcal{O}}$ and $\mathcal{O}$, then by (3.3.1), all points in $P(f) \cap J(f)$ are ramified in $\mathcal{O}$.

In order to achieve our goal, we have followed Mihaljević-Brandt's strategy when proving the corresponding statement for strongly subhyperbolic transcendental maps. Compare to [MB12, Propositions 3.2 and 3.4]. The underlying idea is essentially the same as that in Douady and Hubbard's work for subhyperbolic rational maps [DH84, page 22] (see also [Mil06, §19]): the ramification value of each point in $\mathcal{O}$ is defined as a multiple of the local degrees of all points on its backward orbit. See (3.3.2). In particular, with this definition, all postsingular points of f are ramified. Unlike in the polynomial case, both for strongly subhyperbolic and postcritically separated maps, in addition to those in $P(f)$, more ramified points in $\mathcal{O}$ are needed in order to guarantee expansion, i.e., to guarantee that the set $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$ from Definition 3.6 has "enough points". Thus, the set of ramified points of $\mathcal{O}$ will consist of P_J together with a repelling periodic cycle:

Definition and Proposition 3.10 (Dynamically associated orbifolds)**.** *Let f be a strongly postcritically separated map. Then there exist orbifolds $\mathcal{O} := (S, \nu)$ and $\widetilde{\mathcal{O}} := (\tilde{S}, \tilde{\nu})$ with the following properties:*

(a) Either $S = \mathbb{C} = \tilde{S}$ or $cl(\tilde{S}) \subset S = \mathbb{C} \setminus \overline{U}$, where U is a finite union of bounded Jordan domains.

(b) The set of ramified points of $\mathcal{O}$ equals $P_J \cup B$, where B is a periodic cycle in $J(f) \setminus P_J$.

(c) $J(f) \subset \tilde{S} \subseteq S$ and $P_F \cap S = \emptyset$.

(d) $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are hyperbolic orbifolds.

(e) $f \colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map and the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic.

(f) There exists $p \in S \setminus P_J$ such that $\#\left(f^{-1}(p) \cap \tilde{S}\right)$ is infinite and

$$\#\left(\{z \in f^{-1}(p) : \tilde{\nu}(z) \leq \nu(z)\}\right) < \infty.$$

We say that a pair $(\widetilde{\mathcal{O}}, \mathcal{O})$ of Riemann orbifolds is dynamically associated to f *if $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ satisfy (a)-(f).*

Proof. If $F(f) = \emptyset$, then we define $S := \mathbb{C}$. Otherwise, by Lemma 3.5, P_F is contained in a finite union of attracting basins, and so, by Proposition 2.11 we can find bounded Jordan domains $U_1, \dots, U_n$ such that for $U := \cup_{i=1}^n U_i$ it holds that $P_F \cup f(U) \Subset U \Subset F(f)$. We then define $S := \mathbb{C} \setminus \overline{U}$. In particular, S is connected and $J(f) \subset S$. We claim that there exists at least a periodic cycle, that we denote by B, contained in $J(f) \setminus P_J$. This is because since f is an entire transcendental function, $J(f)$ must contain non-degenerate continua [Bak75], $J(f)$ can be characterized as the closure of repelling periodic points [Ber93, Theorem 4], and in addition we have assumed that P_J is discrete. We define the map $\nu\colon S \to \mathbb{N}^+$ as

$$\nu(z) := \begin{cases} \operatorname{lcm}\{\deg(f^m, w), \text{ where } f^m(w) = z \text{ for some } m \geq 1\} & \text{if } z \notin B, \\ 2 & \text{if } z \in B. \end{cases} \tag{3.3.2}$$

Note that no critical point of S belongs to a periodic cycle, since $P_F \subset U$ and by Lemma 3.5, all periodic cycles in $J(f)$ are repelling. By this, Definition 3.3, and expanding the definition of local degree for an iterate of f, there exists a constant C such that for any $w \in S$ and $m \geq 1$,

$$\deg(f^m, w) = \prod_{j=1}^{m} \deg(f, f^j(w)) \leq C. \tag{3.3.3}$$

Therefore, $\nu(z) \leq \operatorname{lcm}\{1, 2, \dots, C\} < \infty$ for all $z \in S$. Moreover, the map ν is defined in (3.3.2) such that $\nu(z) > 1$ if and only if z belongs to $P_J \cup B$. Hence, since f is postcritically separated, P_J is discrete, and thus $\mathcal{O} := (S, \nu)$ is a Riemann orbifold. In particular, by construction item (b) follows.

The orbifold $\mathcal{O}$ is hyperbolic: if $S \neq \mathbb{C}$, then this follows from Theorem 2.4. If on the contrary $S = \mathbb{C}$, by [McM94, Theorem A2], the only orbifolds such that $S = \mathbb{C}$ are either hyperbolic, or they are parabolic with signature (n) or $(2, 2)$. It is shown in [MB12, Proof of Proposition 3.2] that

- for any $n \geq 2$, each orbifold with underlying surface $\mathbb{C}$ and signature (n) must contain an asymptotic value in S, and
- the orbifold with surface $\mathbb{C}$ and signature $(2, 2)$ can only occur for polynomials.

These two cases lead to contradictions with $\operatorname{AV}(f) \cap S = \emptyset$ and f being postcritically separated. Thus, $\mathcal{O}$ must be hyperbolic. By definition of the map ν, for every $z \in f^{-1}(S)$, $\deg(f,z)$ divides $\nu(f(z))$, and hence we can define

$$\tilde{S} \coloneqq f^{-1}(S) \quad \text{and} \quad \tilde{\nu}(z) \colon \tilde{S} \to \mathbb{N}^+ \quad \text{with} \quad \tilde{\nu}(z) \coloneqq \frac{\nu(f(z))}{\deg(f,z)}. \tag{3.3.4}$$

Since the set of ramified points of $\mathcal{O}$ is discrete, one can see, using for example the Identity Theorem, that the set $\{z \in \tilde{S}$ such that $\tilde{\nu}(z) > 1\}$ is also discrete. Thus, $\widetilde{\mathcal{O}} \coloneqq (\tilde{S}, \tilde{\nu})$ is an orbifold. By construction, $\operatorname{AV}(f) \cap S = \emptyset$, and so the map $f \colon \tilde{S} \to S$ is a branched covering. Furthermore, for all $z \in \tilde{S}$, $\deg(f,z) \cdot \tilde{\nu}(z) = \nu(f(z))$, and hence $f \colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map.

Recall that $f(U) \Subset U$ whenever $U \neq \emptyset$, which implies $\operatorname{cl}(\tilde{S}) \subset S$. Moreover, if $S = \mathbb{C}$, then $J(f) = \mathbb{C}$ and since $\operatorname{AV}(f) \cap J(f) = \emptyset$ by assumption, then $\tilde{S} = f^{-1}(\mathbb{C}) = \mathbb{C}$ and (a) follows. Moreover, since $J(f)$ is a totally invariant set, $J(f) \subset \tilde{S}$, as stated in (c). Let $z \in \tilde{S}$. The definition of ν together with (3.3.3) imply that $\nu(z) \cdot \deg(f,z)$ divides $\nu(f(z))$. In turn, by (3.3.4), $\nu(f(z)) = \tilde{\nu}(z) \cdot \deg(f,z)$. Hence, $\nu(z)$ divides $\tilde{\nu}(z)$ and so the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is a holomorphic map, proving statement (e). Since in addition $\mathcal{O}$ is hyperbolic, by Theorem 2.4 each connected component of $\widetilde{\mathcal{O}}$ must be hyperbolic, and so $\widetilde{\mathcal{O}}$ is a hyperbolic orbifold. Thus, statement (d) follows. We are only left to show item (f). With that purpose, choose any $p \in B$. In particular $p \in S$, and so $f^{-1}(p) \subset \tilde{S}$. Moreover, since $\operatorname{AV}(f) \cap J(f) = \emptyset$, by Picard's theorem, $\# f^{-1}(p)$ is infinite, see [Sch10, Theorem 1.14]. Since $p \in B \subset J(f) \setminus P_J$, $\deg(f,z) = 1$ for all $z \in \operatorname{Orb}^-(p)$ and in particular, for all $z \in f^{-1}(p) \setminus B$, it holds that $\tilde{\nu}(z) = 2$ and $\nu(z) = 1$. Consequently, (f) follows and the proof is concluded. □

Note that condition (f) in the previous proposition implies that for any pair $(\widetilde{\mathcal{O}}, \mathcal{O})$ of orbifolds associated to f, the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is not an orbifold covering map, and hence the set $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$ is non-empty. The next proposition tells us that when in addition $f \in \mathcal{B}$, the set $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$ contains a sequence of points whose moduli converge to infinity at a specified rate.

Proposition 3.11 (Unbounded sequence in $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$)**.** *Let $f \in \mathcal{B}$ be strongly postcritically separated and let $(\widetilde{\mathcal{O}}, \mathcal{O})$ be a pair of orbifolds dynamically associated to f. Then, there exist a constant $N > 1$ and an infinite sequence of points $\{z_i\}_{i \geq 0} \subset \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$ such that $|z_i| < |z_{i+1}| \leq N|z_i|$ for all $i \geq 0$.*

Proof. Let $\mathcal{O} = (S, \nu)$ and let $p \in S \setminus S(f)$ be the point in Proposition 3.10 for which $\#\{z \in f^{-1}(p) : \tilde{\nu}(z) \leq \nu(z)\} < \infty$. That is, all but finitely many preimages of p belong to $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$. Since $f \in \mathcal{B}$, we can find a Jordan domain D such that $S(f) \subset D$ and $p \in \mathbb{C} \setminus \overline{D}$. Since p is non-exceptional, each connected component of $f^{-1}(\mathbb{C} \setminus \overline{D})$, that is, each tract of f, contains infinitely many preimages of p, see Proposition 2.19(2). If $\{z_i\}_{i \geq 0}$ is the set of preimages of p in one tract, it follows from estimates on the hyperbolic metric on simply connected domains that there exists a constant $N' > 1$ such that $|z_{i_k}| < |z_{i_{k+1}}| \leq N'|z_{i_k}|$ for an infinite subsequence $\{z_{i_k}\}_{i_k \in \mathbb{N}}$. For details on this argument see [Rem09, Proof of Lemma 5.1] or [MB10, Proof of Proposition 3.4]. Hence, since all but finitely many points of $\{z_i\}_i$ must belong to $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$, the statement follows. $\square$

Note that Corollary 2.9 applies to any pair of orbifolds $(\widetilde{\mathcal{O}}, \mathcal{O})$ dynamically associated to f, and so $\|Df(z)\|_{\mathcal{O}} = \rho_{\widetilde{\mathcal{O}}}(z)/\rho_{\mathcal{O}}(z) > 1$ for all unramified $z \in \widetilde{\mathcal{O}}$. We aim to prove Theorem 1.1 for any such pair of associated orbifolds by finding a sharper uniform lower bound for $\rho_{\widetilde{\mathcal{O}}}/\rho_{\mathcal{O}}$ combining the following lemma with Theorem 1.12. In turn, Lemma 3.12 is a consequence of Proposition 3.11 together with Theorem 1.13 and item (c) in the Definition 3.3 of strongly postcritically separated maps:

Lemma 3.12 (Distances within annuli are uniformly bounded)**.** *Suppose that f is a strongly postcritically separated function with parameters (c, ϵ), and let $\mathcal{O} = (S, \nu)$ and $\widetilde{\mathcal{O}} = (\tilde{S}, \tilde{\nu})$ be a pair of orbifolds dynamically associated to f. Let us fix some constant $K > 1$. Then, there exists a constant $R := R(K) > 0$ such that if $p, q \in \overline{A(t, Kt)} \subset A(t/K, tK^2) \subset S$ for some $t > 0$, then $d_{\mathcal{O}}(p, q) \leq R$. If in addition $f \in \mathcal{B}$, then for all $z \in \widetilde{\mathcal{O}}$,*

$$d_{\mathcal{O}}(z, \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}) \leq R.$$

Proof. By Proposition 3.10(b), the set of ramified points of $\mathcal{O}$ equals $P_J \cup B$, where B is a periodic cycle in $J(f) \setminus P_J$. Thus, since f is strongly postcritically separated, by Observation 3.4, there exists a constant $M > 0$ so that for each $r > 0$ such that $\overline{A(r, Kr)} \subset \mathcal{O}$, the closed annulus $\overline{A(r, Kr)}$ contains at most $\tilde{M} := M + \#B$ ramified points of $\mathcal{O}$. For each $b \in B$, let $\epsilon_b > 0$ be such that

$$\text{if } z, w \in \left((P_J \cup B) \cap \overline{A(K^{-1}|b|, K|b|)}\right), \qquad \text{then } |z - w| \geq \epsilon_b \max\{|z|, |w|\}. \tag{3.3.5}$$

For each $b \in B$, the constant ϵ_b exists because $\#B$ is finite and P_J is discrete. Note that if $b \in \overline{A(r, Kr)} \subset \mathcal{O}$ for some $r > 0$, then $\overline{A(r, Kr)} \subset \overline{A(K^{-1}|b|, K|b|)}$. Let

$$\tilde{\epsilon} := \min \left\{ \epsilon, \min_{b \in B} \epsilon_b \right\}.$$

Recall that, by Proposition 3.10(a), S is a punctured neighbourhood of infinity, and so we can fix an arbitrary $r > 0$ such that $A_r := A(r/K, K^2 r) \subset S$. Since

$$\overline{A_r} = \bigcup_{j=0}^{2} \overline{A(K^{j-1}r, K^j r)}, \tag{3.3.6}$$

A_r contains at most $3\tilde{M}$ ramified points of $\mathcal{O}$. Without loss of generality we might assume that $r = 1$, since otherwise the same argument applies by scaling by r. Let $C := \max_{z\in S} \nu(z)$, and note that by (3.3.3), $C < \infty$. Then, by Theorem 1.13 applied to the domain $A(1/K, K^2)$, the compact set $\overline{A(1, K)}$ and the parameters $3\tilde{M}, C \in \mathbb{N}$ and $\tilde{\epsilon} > 0$, we conclude that there exists a constant R_1 such that

$$d_{\hat{\mathcal{O}}}(p, q) < R_1 \quad \text{ for all } \; p, q \in \overline{A(1, K)} \; \text{ and all orbifolds } \; \hat{\mathcal{O}} := (A(1/K, K^2), \nu_{\hat{\mathcal{O}}}), \tag{3.3.7}$$

where $\nu_{\hat{\mathcal{O}}}$ is any ramification map that only assumes values smaller or equal to C, and it does so for at most $3\tilde{M}$ points, that are at Euclidean distance at least $\tilde{\epsilon}$ from each other. We shall now complete the proof of the first part of the statement using (3.3.7): for each $t > 0$ such that $A_t = A(t/K, K^2 t) \subset S$, define the orbifolds $\mathcal{O}_t := (A_t, \nu|_{A_t})$ and $\mathcal{O}_1^t := (A_1, \nu_1^t)$, where $\nu|_{A_t}$ is the restriction of the ramification map ν of $\mathcal{O}$ to A_t, and $\nu_1^t(z) := \nu|_{A_t}(tz)$. Note that by (3.3.6), the definition of ν_1^t, (3.3.5) and Observation 3.4, both $\mathcal{O}_t$, $\mathcal{O}_1^t$ contain at most $3\tilde{M}$ ramified points, any pair at (Euclidean) distance at least $\tilde{\epsilon}$. Consequently, (3.3.7) applies to $\mathcal{O}_1^t$. Then, the map $\varphi_t \colon \mathcal{O}_1^t \to \mathcal{O}_t$ given by $\varphi_t(z) := tz$ is an orbifold covering map, and since $\mathcal{O}_t \hookrightarrow \mathcal{O}$ is holomorphic, by definition of $\mathcal{O}_t$, Corollary 2.7 and (3.3.7), for every $p, q \in \overline{A(t, Kt)}$,

$$d_{\mathcal{O}}(p, q) \leq d_{\mathcal{O}_t}(p, q) = d_{\mathcal{O}_1^t}(\varphi_t^{-1}(p), \varphi_t^{-1}(q)) < R_1,$$

and the first statement of the lemma is proved.

In order to prove the second part of the lemma, if $f \in \mathcal{B}$, let $\{z_i\}_{i\geq 0} \subset \mathbb{B}^{\mathcal{O}}_{\tilde{\mathcal{O}}}$ be the infinite sequence of points from Proposition 3.11 for which there exists $N > 1$ such that $|z_i| < |z_{i+1}| \leq N|z_i|$ holds for all $i \geq 0$. Recall that by Proposition 3.10, $S = \mathbb{C} = \tilde{S}$, or S is the complement of a finite union of bounded Jordan domains and $\overline{\tilde{S}} \subset S$. Then there exists a finite number

$$I := \min\left\{ j \geq 0 : \mathbb{C} \setminus \mathbb{D}_{\frac{|z_j|}{K}} \subset S \right\},$$

that equals 0 when $S = \mathbb{C}$. Let $J := \left\lceil \frac{\log N}{\log K} \right\rceil$ and for each $i > I$, denote

$$A_i := \overline{A(|z_{i-1}|, N|z_{i-1}|)} \subseteq \bigcup_{j=1}^{J} \overline{A\left(K^{j-1}|z_{i-1}|, K^j|z_{i-1}|\right)}.$$

In particular, $z_i \in A_i$ for all $i > I$. Hence, for all $z \in A_i \subset S$, using the first part of the lemma, $d_{\mathcal{O}}(z, \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}) \leq J \cdot R_1 =: R_2$. Since the constant J is independent of the index $i > I$, we can conclude that

$$d_{\mathcal{O}}(z, \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}) < R_2 \quad \text{ for all } \quad z \in \bigcup_{i>I} A_i = \mathbb{C} \setminus \mathbb{D}_{|z_I|}. \tag{3.3.8}$$

If $\tilde{S} \subset \mathbb{C} \setminus \mathbb{D}_{|z_I|}$, we are done. Otherwise, recall that either $\mathbb{C} \setminus \mathbb{D}_{|z_I|} \subset \tilde{S} = \mathbb{C}$, or we have that $\operatorname{cl}(\tilde{S}) \subset S$. In any case, we can consider the compact set $\mathcal{K} := \operatorname{cl}(\mathbb{D}_{|z_I|} \cap \tilde{S})$ and any domain U such that $\mathcal{K} \subset U \subset S$. In particular, if $\mathcal{O}_U := (U, \nu|_U)$, then the inclusion $\mathcal{O}_U \hookrightarrow \mathcal{O}$ is holomorphic. Note also that in the first case, $z_I \in \mathcal{K}$, while in the second, $\partial S \cap \mathcal{K} \neq \emptyset$ and all points in that intersection also belong to $\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$. Consequently, in any case we can choose a point $p \in \mathcal{K} \cap \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}$. Let $\tilde{N}$ be the number of ramified points of $\mathcal{O}_U$. If $\tilde{N} > 1$, let δ be the minimum of the (Euclidean) distances between any two ramified points in U. Otherwise, if $\tilde{N}$ equals 0 or 1, let δ be any real positive number. Then, by Corollary 2.7 and Theorem 1.13 applied to $U, \mathcal{K}$ and the parameters $C, \tilde{N} \in \mathbb{N}$ and $\delta > 0$, there exists a constant $R_3 > 0$ such that for all $z \in \operatorname{cl}(\mathbb{D}_{|z_I|} \cap \tilde{S})$,

$$d_{\mathcal{O}}(z, \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}) \leq d_{\mathcal{O}_U}(z, \mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}) \leq d_{\mathcal{O}_U}(z, p) < R_3.$$

By this together with (3.3.8), the lemma follows letting $R := \max\{R_1, R_2, R_3\}$. $\square$

Theorem 1.1 now follows easily on combining Lemma 3.12 and Theorem 1.12:

Proof of Theorem 1.1. Let $f \in \mathcal{B}$ strongly postcritically separated. By Proposition 3.10, there exists a pair of hyperbolic orbifolds $\mathcal{O} := (S, \nu)$ and $\widetilde{\mathcal{O}} := (\tilde{S}, \tilde{\nu})$ such that $J(f) \subset S \cap \tilde{S}$, $f\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is a covering map and the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic. Hence, by Corollary 2.9,

$$\|Df(z)\|_{\mathcal{O}} = \frac{|f'(z)|\rho_{\mathcal{O}}(f(z))}{\rho_{\mathcal{O}}(z)} = \frac{\rho_{\widetilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)}. \tag{3.3.9}$$

Moreover, by Lemma 3.12, there exists a constant R such that $d_{\mathcal{O}}(\mathbb{B}^{\mathcal{O}}_{\widetilde{\mathcal{O}}}, z) < R$ for all unramified $z \in \widetilde{\mathcal{O}}$. Thus, by Theorem 1.12 and using (3.3.9), for all unramified $z \in \widetilde{\mathcal{O}}$, $\|Df(z)\|_{\mathcal{O}} \geq (\mathrm{e}^R / \sqrt{\mathrm{e}^{2R} - 1}) =: \Lambda > 1$, as we wanted to show. $\square$

As a consequence of Theorem 1.1, we obtain the following corollary that relates the $\mathcal{O}$-length of bounded curves to the $\mathcal{O}$-length of its successive images.

Corollary 3.13 (Shrinking of preimages of bounded curves)**.** *Let $f \in \mathcal{B}$ be a strongly postcritically separated map, and let $(\widetilde{\mathcal{O}}, \mathcal{O})$ be a pair of dynamically associated orbifolds. Then, for any curve $\gamma_0 \subset \mathcal{O}$, for all $k \geq 1$ and each curve $\gamma_k \subset f^{-k}(\gamma_0)$ such that $f^k|_{\gamma_k}$ is injective,*

$$\ell_{\mathcal{O}}(\gamma_k) \leq \frac{\ell_{\mathcal{O}}(\gamma_0)}{\Lambda^k}$$

for some constant $\Lambda > 1$.

Proof. By Theorem 1.1 and Corollary 2.9, there exists a constant Λ such that for all unramified $z \in \widetilde{\mathcal{O}}$,

$$\|Df(z)\|_{\mathcal{O}} = \frac{\rho_{\widetilde{\mathcal{O}}}(z)}{\rho_{\mathcal{O}}(z)} \geq \Lambda > 1. \tag{3.3.10}$$

In particular, recall that the set of ramified points in $\widetilde{\mathcal{O}}$ is negligible when computing the length of bounded curves, as it is discrete and so has Lebesgue measure 0. Let γ_0 be any curve as in the statement. We proceed by induction on k. Suppose $k = 1$ and let us parametrize the curves γ_0 and γ_1 such that $f(\gamma_1(t)) = \gamma_0(t)$ for all $t \geq 0$. Since by Proposition 3.10 $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map, by Theorem 2.6, $\rho_{\widetilde{\mathcal{O}}}(\gamma_1(t)) = |f'(\gamma_1(t))| \cdot \rho_{\mathcal{O}}(\gamma_0(t))$ for all $t \geq 0$. Using this and (3.3.10),

$$\begin{aligned}\ell_{\mathcal{O}}(\gamma_1) &= \int |\gamma_1'(t)| \rho_{\mathcal{O}}(\gamma_1(t)) dt = \int |\gamma_1'(t)| \frac{\rho_{\mathcal{O}}(\gamma_1(t))}{\rho_{\widetilde{\mathcal{O}}}(\gamma_1(t))} \rho_{\widetilde{\mathcal{O}}}(\gamma_1(t)) dt \\ &\leq \frac{1}{\Lambda} \int |\gamma_1'(t)| \cdot |f'(\gamma_1(t))| \rho_{\mathcal{O}}(\gamma_0(t)) dt = \frac{1}{\Lambda} \int |\gamma_0'(t)| \rho_{\mathcal{O}}(\gamma_0(t)) dt \leq \frac{\ell_{\mathcal{O}}(\gamma_0)}{\Lambda}.\end{aligned}$$

Let us suppose that the statement is true for some $k-1$. Then, if $\gamma_k \in f^{-k}(\gamma_0)$, $f(\gamma_k) = \gamma_{k-1}$ for some curve $\gamma_{k-1} \in f^{-k+1}(\gamma_0)$. By the same argument as before and using the inductive hypothesis,

$$\ell_{\mathcal{O}}(\gamma_k) \leq \frac{1}{\Lambda} \int |\gamma_{k-1}'(t)| \rho_{\mathcal{O}}(\gamma_{k-1}(t)) dt = \frac{1}{\Lambda} \ell_{\mathcal{O}}(\gamma_{k-1}) \leq \frac{\ell_{\mathcal{O}}(\gamma_0)}{\Lambda^k}. \qquad \square$$

3.4 Results on the topology of Fatou and Julia sets

In this section we provide the proofs of Theorem 1.2 and Corollaries 1.3 and 1.4. We note that the arguments in the proofs of the corresponding results for hyperbolic maps in [BFRG15], rely mostly in the maps being *expanding*. That is, in their derivative with respect to the hyperbolic metric being greater than one in a punctured neighbourhood of infinity that contains their Julia set. Since we have achieved an analogous result for strongly postcritically separated maps in Theorem 1.1, we are able to adapt most of the proofs in [BFRG15] with few modifications. We start by borrowing some auxiliary results from [BFRG15]. The first one is a well-known result that we cite as stated in [BFRG15, Lemma 2.7].

Lemma 3.14 (Coverings of doubly-connected domains)**.** *Let $U, V \subset \mathbb{C}$ be domains and let $f\colon V \to U$ be a covering map. Suppose that U is doubly-connected. Then either V is doubly-connected and f is a proper map, or V is simply connected and f is a universal cover of infinite degree.*

The next proposition gathers some well-known facts on the behaviour of entire maps on preimages of simply-connected domains. For ease of reference, the following statement merges [BFRG15, Propositions 2.8 and 2.9]. For the first part, compare to [Hei57, Her98, Bol99].

Proposition 3.15 (Mapping of simply connected sets)**.** *Let f be an entire function, let $D \subset \mathbb{C}$ be a simply connected domain, and let $\widetilde{D}$ be a component of $f^{-1}(D)$. Then either*

(1) $f\colon \widetilde{D} \to D$ is a proper map and hence has finite degree, or

(2) for every $w \in D$ with at most one exception, $\#(f^{-1}(w) \cap \widetilde{D})$ is infinite. In this case, either $\widetilde{D}$ contains an asymptotic curve corresponding to an asymptotic value in D, or $\widetilde{D}$ contains infinitely many critical points.

If in addition $D \cap S(f)$ is compact,

(A) If $\#(D \cap S(f)) \leq 1$, then $\widetilde{D}$ contains at most one critical point of f.

(B) In case (1), if D is a bounded Jordan domain such that $\partial D \cap S(f) = \emptyset$, then $\widetilde{D}$ is also a bounded Jordan domain.

(C) In case (2), the point ∞ is accessible from $\widetilde{D}$.

In addition, we will make use of the following result in order to show that the boundaries of certain Fatou components are not locally connected.

Theorem 3.16 (Boundaries of periodic Fatou components [BFRG15, Theorem 2.6])**.** *Let f be a transcendental entire function, and suppose that U is an unbounded periodic component of $F(f)$ such that $f^n|_U$ does not tend to infinity. Then $\widehat{\mathbb{C}} \setminus U$ is not locally connected at any finite point of ∂U.*

The proof of Theorem 1.2 will follow easily once we show that whenever condition (b) on its statement holds, every *periodic* Fatou component is bounded. We achieve so in the following theorem. In particular, we note the similarities with [BFRG15, Theorem 1.10]: Theorem 3.17 holds for a more general class of maps, but [BFRG15, Theorem 1.10] has the stronger conclusion that periodic Fatou components are quasidiscs. We suspect that this is also the case for the class of maps we study. However, we have not been able to conclude so; see §5.2 for further discussion.

Theorem 3.17 (Immediate basins of strongly postcritically separated maps)**.** *Let $f \in \mathcal{B}$ be strongly postcritically separated and let D be a periodic Fatou component of f, of some period $p \geq 1$. Then the following are equivalent:*

(1) D is a Jordan domain;

(2) $\widehat{\mathbb{C}} \setminus D$ is locally connected at some finite point of ∂D;

(3) D is bounded;

(4) the point ∞ is not accessible in D;

(5) the orbit of D contains no asymptotic curves and only finitely many critical points;

(6) $f^p : D \to D$ is a proper map;

(7) for at least two distinct choices of $z \in D$, the set $f^{-p}(z) \cap D$ is finite.

Proof. Let f and D be as in the statement. In particular, D is simply connected (multiply-connected Fatou components of transcendental entire functions are wandering domains [Bak84, Theorem 3.1]). By passing to an iterate, we may assume without loss of generality that $p = 1$. Since the complement of a Jordan domain is locally connected at every point, (1)⇒(2) is immediate. If (2) holds, then since by Lemma 3.5 all Fatou components of f belong to attracting cycles, by Theorem 3.16 D must be bounded, and so (2) implies (3). If D is bounded, then D cannot contain a curve to ∞, and hence (3)⇒(4). Since f is postcritically separated and $D \subset F(f)$, $P(f) \cap D$ is compact. Thus, by Proposition 3.15(C), if infinity is not accessible in D, then item (1) must occur in Proposition 3.15, and so D contains only finitely many critical points and no asymptotic values, which is equivalent to $f : D \to D$ being a proper map. Thus, (4)⇒(5)⇔(6). Since any proper map has finite degree, (6)⇒(7).

To conclude the proof it suffices to show that (7)⇒(1). With that aim, suppose that (7) holds for f. Recall that by Proposition 2.11, there exists a bounded Jordan domain $U_0 \Subset D$ such that $\overline{f(U_0)} \subset U_0$ and $P(f) \cap D \Subset U_0$. For each $n \geq 1$, let

$$U_n := f^{-n}(U_0) \cap D,$$

and note that by the property $\overline{f(U_0)} \subset U_0$, one can see using induction that

$$U_n \subset U_{n+1} \quad \text{for all} \quad n \geq 0, \quad \text{and} \quad D = \bigcup_{n=0}^{\infty} U_n. \tag{3.4.1}$$

Since we have assumed that (7) holds for f, so does Proposition 3.15(1), and hence $f\colon D \to D$ is a proper map of some degree $d \geq 1$. Moreover, by definition of U_0, for each $n \geq 1$, $f^n\colon D \setminus \overline{U_n} \to A := D \setminus \overline{U_0}$ is a finite-degree covering map (of degree d^n) over the doubly-connected domain A. By Lemma 3.14, the domain $D \setminus \overline{U_n}$ is also doubly-connected, and hence U_n is connected for all n. Furthermore, since $P(f) \cap D \Subset U_0$, it follows from Proposition 3.15(B) applied to $f^n\colon U_n \to U_0$ that each U_n is a bounded Jordan domain. Hence, for every $n \geq 0$, $f\colon \partial U_{n+1} \to \partial U_n$ is, topologically, a covering map of degree d over a circle.

Claim. There exists a diffeomorphism $\varphi\colon \{z \in \mathbb{C}\colon 1/e < |z| < 1\} \to D \setminus \overline{U_0}$ such that

$$f(\varphi(z)) = \varphi(z^d) \qquad \text{whenever } e^{-1/d} < |z| < 1. \tag{3.4.2}$$

This claim and its proof appear in [BFRG15, Proof of Theorem 1.10], and thus we omit the proof. Our next and final goal is to extend continuously the domain of the function φ to include $\partial\mathbb{D}$. With that aim, for each $\theta \in \mathbb{R}$ and $n \geq 0$, consider the curve

$$\gamma_{n,\theta} := \varphi(\{e^{a+i\theta}\colon\ -d^{-n} \leq a \leq -d^{-(n+1)}\}). \tag{3.4.3}$$

Note that by the commutative relation in (3.4.2), $\gamma_{n,\theta}$ is the preimage of the arc $\gamma_{0,\theta\cdot d^n}$ under some branch of f^{-n}, and in particular is a simple curve with end-points in ∂U_n and ∂U_{n+1}.

Let $\widetilde{\mathcal{O}} := (\widetilde{S}, \tilde{\nu})$ and $\mathcal{O} := (S, \nu)$ be a pair of orbifolds dynamically associated to f. In particular, by Proposition 3.10(a), the underlying surface S of $\mathcal{O}$ can be chosen so that $D \setminus U_0 \subset S$. Note that for each $\tilde{\theta} \in \mathbb{R}$, the curve $\gamma_{0,\tilde{\theta}}$ is contained in the compact set $\overline{U_1 \setminus U_0}$. Since $D \setminus U_0 \subset F(f) \setminus P(f)$, by Proposition 3.10(b) there are no ramified points of $\mathcal{O}$ in $\overline{U_1 \setminus U_0}$. Thus, the orbifold metric $\rho_{\mathcal{O}}$ attains a maximum value in $\overline{U_1 \setminus U_0}$, and since the Euclidean length of the curves $\{\gamma_{0,\tilde{\theta}}\}_{\tilde{\theta}}$

must be finite, as these curves are the image under φ of a straight line, there exists a constant $L > 0$ such that

$$\max_{\widetilde{\theta}} \ell_{\mathcal{O}}(\gamma_{0,\widetilde{\theta}}) < L.$$

Moreover, for all $n \geq 0$ and $\theta \in \mathbb{R}$, the curve $\gamma_{n,\theta} \subset D \setminus U_0 \subset S \setminus P(f)$, and so, since D is by assumption invariant, f^n maps $\gamma_{n,\theta}$ injectively to $\gamma_{0,\theta \cdot d^n}$. Hence, we can apply Corollary 3.13 to conclude that there exists a constant $\Lambda > 1$ such that

$$\ell_{\mathcal{O}}(\gamma_{n,\theta}) \leq \frac{\ell_{\mathcal{O}}(\gamma_{0,\theta \cdot d^n})}{\Lambda^n} \leq \frac{\max_{\widetilde{\theta}} \ell_{\mathcal{O}}(\gamma_{0,\widetilde{\theta}})}{\Lambda^n} \leq \frac{L}{\Lambda^n}, \tag{3.4.4}$$

where we note that the upper bound is independent of θ. For each $n \geq 0$, let us define the function

$$\sigma_n \colon \mathbb{R}/\mathbb{Z} \to \partial U_n \quad \text{as} \quad \sigma_n(t) := \varphi\left(e^{-d^{-n}+2\pi t i}\right).$$

Note that for each $t \in \mathbb{R}/\mathbb{Z}$, the curve $\gamma_{n,2\pi t}$ defined in (3.4.3) joins $\sigma_n(t)$ and $\sigma_{n+1}(t)$. Thus, by (3.4.4), $\{\sigma_n\}_n$ forms a Cauchy sequence of continuous functions. Consequently, using (3.4.1), there exists a limit function $\sigma \colon \partial\mathbb{D} \to \partial D$, which by (3.4.2) is the continuous extension of φ to the unit circle. Hence, ∂D is a continuous closed curve as it is the continuous image of $\partial\mathbb{D}$. In particular, D is bounded. By the maximum principle, $\partial D = \partial\overline{D}$ and $\mathbb{C} \setminus \overline{D}$ has no bounded connected components. Hence D is a Jordan domain. □

Using the preceding theorem, we are now ready to provide the proofs of our results on the topology of Fatou and Julia sets.

Proof of Theorem 1.2. We start proving that (a) implies (b) by showing the contrapositive. Note that since f is strongly postcritically separated, all asymptotic values of f must lie in $F(f)$, and hence if $\mathrm{AV}(f) \neq \emptyset$, then $F(f)$ must have an unbounded component by definition of asymptotic value. Moreover, if some Fatou component U contains infinitely many critical points, since these are the zeros of the analytic function f', they can only accumulate at infinity and therefore U is unbounded.

To prove that (b) implies (a), we note that by Lemma 3.5, all Fatou components of f are (pre)periodic. If (b) holds for f, that is, $\mathrm{AV}(f) = \emptyset$ and each Fatou component contains at most finitely many critical points, then by Theorem 3.17 ((5)$\Longleftrightarrow$(1)), every periodic Fatou component is a bounded Jordan domain. Next we see that strictly preperiodic Fatou components are also bounded Jordan domains. If V is any preimage of a periodic Fatou component U, then, by assumption, Proposition 3.15(2) cannot hold. Thus, $f\colon V \to U$ must be a proper map. In addition, V is also bounded by Proposition 3.15(B). Proceeding by induction on the pre-period of V, the claim follows. $\square$

In order to prove Corollary 1.3, we will make use of a result from [BM02], where the concept of *semihyperbolic* entire maps is introduced:

Definition 3.18 (Semihyperbolic functions)**.** An entire function f is *semihyperbolic at a point* p if there exist $r > 0$ and $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all components U of $f^{-n}(\mathbb{D}_r(p)) = \{z \in \mathbb{C} : f^n(z) \in \mathbb{D}_r(p)\}$, the function $f^n|_U \colon U \to \mathbb{D}_r(p)$ is a proper map of degree at most N. A function f is *semihyperbolic* if f is semihyperbolic at all $p \in J(f)$.

Proposition 3.19. *If f is strongly postcritically separated, then f is semihyperbolic.*

Proof. Let us fix $p \in J(f)$. Since $P(f) \cap J(f)$ is discrete and $P(f) \cap F(f)$ is compact, there exists $r > 0$ such that $\mathbb{D}_r(p) \cap P(f)$ contains at most the point p. By Definition 3.3, there exist constants $C, \mu > 0$ such that for all $z \in J(f)$,

$$\#(\mathrm{Orb}^+(z) \cap \mathrm{Crit}(f)) \leq C \quad \text{and} \quad \deg(f, z) < \mu.$$

Therefore, for any $n \geq 0$ and any connected component U of $f^{-n}(\mathbb{D}_r(p))$, since $U \cap \mathrm{Orb}^-(P(f)) \subset \mathrm{Orb}^-(p)$, $f^n|_U$ is a proper map of degree at most $\mu^C =: N$. $\square$

The following theorem is a version of [BM02, Theorem 4] for our class of maps. In particular, this theorem tells us that if Fatou components are Jordan domains, in certain cases local connectivity of their Julia sets follows.

Theorem 3.20 (Bounded components and bounded degree imply local connectivity)**.** *Let $f \in \mathcal{B}$ be strongly postcritically separated with no asymptotic values. Suppose that every immediate attracting basin of f is a Jordan domain. If there exists $N \in \mathbb{N}$ such that the degree of the restriction of f to any Fatou component is bounded by N, then $J(f)$ is locally connected.*

Remark. We note that [BFRG15, Theorem 2.5] is a version of Theorem 3.20 for hyperbolic maps whose proof is based on expansion of hyperbolic maps in a neighbourhood of their Julia set. Therefore and alternatively, we could have presented an analogous proof for functions in class $\mathcal{B}$ that are strongly postcritically separated using Theorem 1.1 in a similar manner as we did in the proof of Theorem 3.17.

Proof of Corollary 1.3. Let $f \in \mathcal{B}$ be strongly postcritically separated with no asymptotic values, and assume that every Fatou component of f contains at most N critical points, counting multiplicity, for some $N \in \mathbb{N}$. Then, hypothesis (b) in Theorem 1.2 holds for f, and consequently every Fatou component U is a bounded Jordan domain. Moreover, Proposition 3.15(1) must hold and so the restriction $f|_U : U \to f(U)$ is a proper map. Since f has no wandering domains (Lemma 3.5), all Fatou components of f are simply connected [Bak84, Theorem 3.1]. Then, the Riemann-Hurwitz formula, see [Mil06, Theorem 7.2], tells us that the degree of $f|_U$ is bounded by $N+1$. Consequently, local connectivity of $J(f)$ follows from Theorem 3.20. □

Proof of Corollary 1.4. Let f be strongly postcritically separated with no asymptotic values, and assume that every Fatou component contains at most one critical value. Then, by Proposition 3.15(A), each Fatou component also contains at most one critical point. Since by assumption the multiplicity of the critical points is uniformly bounded, local connectivity of $J(f)$ is a consequence of Corollary 1.3. □

3.5 Pullbacks and post-homotopy classes

Given an entire function f and two curves $\gamma, \beta \subset f(\mathbb{C}) \setminus P(f)$ homotopic to each other and with fixed endpoints, by the homotopy lifting property, for each curve in $f^{-1}(\gamma)$ there exists a curve in $f^{-1}(\beta)$ homotopic to it and sharing the same endpoints. In Proposition 3.24 we get an analogue of this result for certain class of curves that contain postsingular points by using a modified notion of homotopy. Moreover, in this section we also show that if f is an entire function with dynamic rays in its Julia set and U is a certain bounded domain of any hyperbolic orbifold whose underlying surface intersects $J(f)$, then there exists a constant μ such that for every piece of dynamic ray contained in U, we can find a curve in its "modified-homotopy" class with orbifold length at most μ. See Corollary 3.27. In particular, this result is crucial in the proof of Theorem 1.8.

For completeness and in order to fix notation, we include some definitions regarding homotopy and covering spaces theory that we require, and we refer the reader to [Hat02, Chapter 1] or [Mun00, Chapter 9] for an introduction to these topics. In this section, by a *curve* in a space X we mean a continuous map $\gamma : I \to X$ with $I = [0,1]$, and in particular its image $\gamma(I)$ is bounded. With slight abuse of notation, we also refer by γ to $\gamma(I)$, and we denote by $\text{int}(\gamma)$ the curve obtained from γ by removing its endpoints. A *homotopy of curves* in X is a family $\{\gamma_t : I \to X\}_{t\in[0,1]}$ for which the associated map $\overline{\gamma} : I \times [0,1] \to X$ given by $\overline{\gamma}(s,t) := \gamma_t(s)$ is continuous. Two curves α and β are said to be *homotopic in* X when there exists a homotopy $\{\gamma_t\}_{t\in[0,1]}$ in X such that $\gamma_0 \equiv \alpha$ and $\gamma_1 \equiv \beta$. Being homotopic is an equivalence relation on the set of all curves in X. Given a covering space $f : \tilde{X} \to X$, a *lift* of a map $g : Y \to X$ by f is a map $\tilde{g} : Y \to \tilde{X}$ such that $f \circ \tilde{g} = g$. The main result that serves our purposes is the following:

Proposition 3.21 (Homotopy lifting property)**.** *Given a covering space $f : \tilde{X} \to X$, a homotopy $\{\gamma_t : Y \to X\}_{t\in[0,1]}$ and a map $\tilde{\gamma}_0 : Y \to \tilde{X}$ lifting γ_0, there exists a unique homotopy $\{\tilde{\gamma}_t : Y \to \tilde{X}\}_{t\in[0,1]}$ that lifts $\{\gamma_t\}_{t\in[0,1]}$.*

Proof. See [Hat02, Proposition 1.30] for the proof of the statement whenever the homotopies have fixed endpoints, and [GH81, (5.3) Covering Homotopy Theorem] or [Hat02, Section 4.2] for the general case. □

Recall that for an entire function f, its singular set $S(f)$ is the smallest closed set for which $f : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f)$ is a covering map, and regarding the iterates of f, recall that by Proposition 2.13, for each $k \geq 1$, $S(f^k) \subseteq P(f)$. Consequently, for all $k \geq 1$ and an entire function f,

$$f^k\colon \mathbb{C} \setminus f^{-k}(P(f)) \to \mathbb{C} \setminus P(f) \ \text{ is a covering map.} \tag{3.5.1}$$

Thus, the homotopy lifting property applies to any homotopy of curves in $\mathbb{C}\setminus P(f)$. We are interested in obtaining an analogous property that applies to certain curves whose image in $\mathbb{C}$ contains postsingular points. We specify now which curves we are interested in:

3.22 (Definition of the sets $\mathcal{H}_p^q(W(k))$)**.** Let us fix an entire function f and let $k \in \mathbb{N}$. We suggest the reader keeps in mind the case when $k = 0$, since it will be the one of greatest interest for us. Let $W(k)$ be a finite set of (distinct) points in $f^{-k}(P(f))$, totally ordered with respect to some relation "$\prec$". That is, $W(k) := (W(k), \prec) = \{w_1, \dots, w_N\} \subset f^{-k}(P(f))$ such that $w_{j-1} \prec w_j \prec w_{j+1}$ for all $1 < j < N$. We note that $W(k)$ can be the empty set. Then, for every pair of points[5] $p, q \in \mathbb{C} \setminus W(k)$, we denote by $\mathcal{H}_p^q(W(k))$ the collection of all curves in $\mathbb{C}$ with endpoints p and q that join the points in $W(k)$ *in the order* "$\prec$" starting from p. More formally, $\gamma \in \mathcal{H}_p^q(W(k))$ if $\operatorname{int}(\gamma) \cap f^{-k}(P(f)) = W(k)$ and γ can be parametrized so that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma(\frac{j}{N+1}) = w_j$ for all $1 \leq j \leq N$. In particular, γ can be expressed as a concatenation of $N + 1$ curves

$$\gamma = \gamma_p^{w_1} \cdot \gamma_{w_1}^{w_2} \cdot \dots \cdot \gamma_{w_N}^{q}, \tag{3.5.2}$$

[5]In particular, p and q might belong to $f^{-k}(P(f))$.

each of them with endpoints in $W(k) \cup \{p, q\}$ and such that

$$\mathrm{int}(\gamma_p^{w_1}), \mathrm{int}(\gamma_{w_i}^{w_{i+1}}), \mathrm{int}(\gamma_{w_N}^{q}) \in \mathbb{C} \setminus f^{-k}(P(f))$$

for each $1 \leq i \leq N$. See Figure 3.

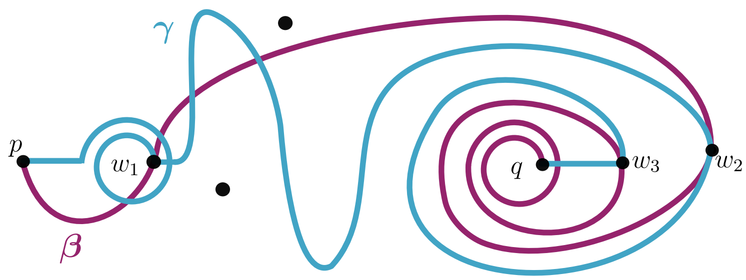

Figure 3: Example of two curves $\gamma, \beta \in H_p^q(\{w_1, w_2, w_3\})$ that are post-k-homotopic for some $k \geq 1$. Points in $f^{-k}(P(f))$ are represented by black dots.

We use a modified notion of *homotopy* for the sets of curves described:

Definition 3.23 (Post-k-homotopic curves)**.** Consider $W(k) = \{w_1, \ldots, w_N\} \subset f^{-k}(P(f))$ and two curves $\gamma, \beta \in \mathcal{H}_{w_0}^{w_{N+1}}(W(k))$, for some $\{w_0, w_{N+1}\} \subset \mathbb{C} \setminus W(k)$. We say that γ is *post-k-homotopic to* β if for all $0 \leq i \leq N$, $\gamma_{w_i}^{w_{i+1}}$ is homotopic to $\beta_{w_i}^{w_{i+1}}$ in $(\mathbb{C} \setminus f^{-k}(P(f))) \cup \{w_i, w_{i+1}\}$.

In particular, for each $1 \leq i \leq N$ and $k = 0$, the restrictions of γ and β between w_i and w_{i+1} are homotopic in the space $(\mathbb{C} \setminus P(f)) \cup \{w_i, w_{i+1}\}$. See Figure 3. It is easy to see that this defines an equivalence relation in $\mathcal{H}_p^q(W(k))$. For each $\gamma \in \mathcal{H}_p^q(W(k))$, we denote by $[\gamma]_k$ its equivalence class. Note that if $W(k) = \emptyset$ and $p, q \in \mathbb{C} \setminus f^{-k}(P(f))$, then for any curve $\gamma \in \mathcal{H}_p^q(W(k))$, $[\gamma]_k$ equals the equivalence class of γ in $\mathbb{C} \setminus f^{-k}(P(f))$ in the usual sense. Moreover, if γ is any curve that meets only finitely many elements of $f^{-k}(P(f))$, then it belongs to a unique set of the form "$\mathcal{H}_p^q(W(k))$" up to reparametrization of γ, and so its equivalence class $[\gamma]_k$ is defined in an obvious sense. Hence, the notion of post-k-homotopy is well-defined for all such curves, and from now on we will sometimes omit the set of curves they belong to.

The following is an analogue of Proposition 3.21 for post-k-homotopic curves. It's statement is illustrated in Figure 4.

Proposition 3.24 (Post-homotopy lifting property)**.** *Let f be an entire map and let $C \subset \mathbb{C}$ be a domain so that $f^{-1}(C) \subset C$ and $AV(f) \cap C = \emptyset$. Let $\gamma \subset C$ be a bounded curve such that $\#(\gamma \cap P(f)) < \infty$. Fix any $k \geq 0$ and any curve*

$\gamma_k \subset f^{-k}(\gamma)$ for which the restriction $f^k|_{\gamma_k}$ is injective. Then, for each $\beta \in [\gamma]_0$, there exists a unique curve $\beta_k \subset f^{-k}(\beta)$ such that $\beta_k \in [\gamma_k]_k$. In particular, β_k and γ_k share their endpoints.

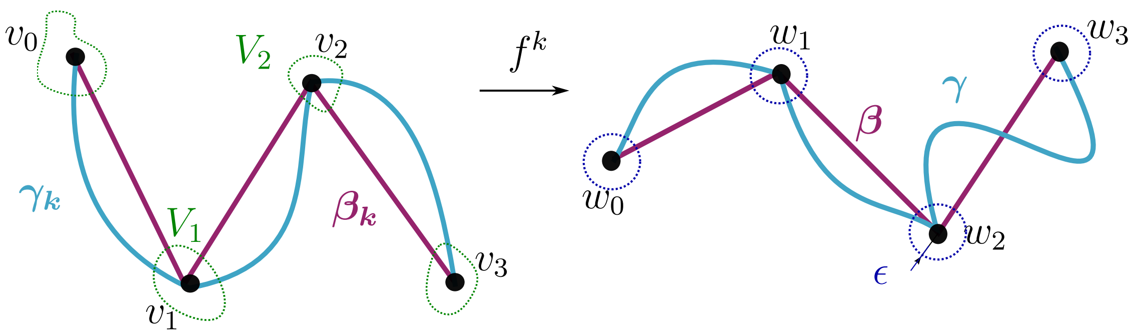

Figure 4: Schematic of some elements appearing in the proof of Proposition 3.24

Proof. Suppose that $\gamma \in \mathcal{H}_{w_0}^{w_{N+1}}(W(0))$, where w_0 and w_{N+1} are the endpoints of γ and $W(0) = P(f) \cap \operatorname{int}(\gamma) =: \{w_1, \dots, w_N\}$ for some $N > 0$. Let

$$\tilde{W}(k) := f^{-k}(P(f)) \cap \operatorname{int}(\gamma_k) = f^{-k}(W(0)) \cap \operatorname{int}(\gamma_k) =: \{v_1, \dots, v_N\}.$$

In particular, $\gamma_k \in \mathcal{H}_{v_0}^{v_{N+1}}(\tilde{W}(k))$ for some $v_0, v_{N+1} \in f^{-k}(\{w_0, w_{N+1}\})$. For each $0 \le i \le N$, we denote by γ_k^i the subcurve in γ_k with endpoints v_i and v_{i+1}.

Similarly, for a fixed $\beta \in [\gamma]_0$ and each $0 \le i \le N$, we denote by β^i and γ^i the respective subcurves in β and γ with endpoints w_i and w_{i+1}. That is, for parametrizations of β and γ such that $\gamma(\frac{i}{N+1}) = w_i = \beta(\frac{i}{N+1})$ for every $0 \le i \le N+1$,

$$\beta^i := \beta|_{[\frac{i}{N}, \frac{i}{N+1}]} \quad \text{and} \quad \gamma^i := \gamma|_{[\frac{i}{N}, \frac{i}{N+1}]}.$$

For each $\epsilon > 0$ small enough, we consider the restrictions $\beta^{i,\epsilon} := \beta|_{[\frac{i}{N}+\epsilon, \frac{i}{N+1}-\epsilon]}$ and $\gamma^{i,\epsilon} := \gamma|_{[\frac{i}{N}+\epsilon, \frac{i}{N+1}-\epsilon]}$. Then, since $\beta^{i,\epsilon} \subset \beta^i$ and $\gamma^{i,\epsilon} \subset \gamma^i$, $\beta^{i,\epsilon}$ is homotopic (in the usual sense) to $\gamma^{i,\epsilon}$ in $(\mathbb{C} \setminus f^{-k}(P(f))) \cup \{w_i, w_{i+1}\}$. Recall that the notion of homotopy does not demand curves to share their endpoints. Therefore, for each $0 \le i \le N$, if $\gamma_k^{i,\epsilon} := \gamma_k^i \cap f^{-k}(\gamma^{i,\epsilon})$, by (3.5.1) and Proposition 3.21, there exists a unique curve $\beta_k^{i,\epsilon} \subset f^{-k}(\beta^{i,\epsilon})$ such that $\beta_k^{i,\epsilon}$ is homotopic to $\gamma_k^{i,\epsilon}$ in $\mathbb{C} \setminus f^{-k}(P(f))$.

We shall now see that as $\epsilon \to 0$, for all $0 \le i \le N$, $\beta_i^{k,\epsilon}$ converges to a curve with endpoints v_i and v_{i+1} homotopic to γ_i^k in $\mathbb{C} \setminus f^{-k}(P(f)) \cup \{v_i, v_{i+1}\}$. Indeed, note that $f^{-k}(w_i)$ and $f^{-k}(w_{i+1})$ are discrete sets of points, and hence we can find open neighbourhoods $V_i \ni v_i$ and $V_{i+1} \ni v_{i+1}$ such that $V_i \cap f^{-k}(w_i) = \{v_i\}$ and $V_{i+1} \cap f^{-k}(w_{i+1}) = \{v_{i+1}\}$. By the assumption $\operatorname{AV}(f) \cap C = \emptyset$, using the Open Mapping theorem, we can conclude that $f^k|_{V_i}$ is an open map, and so, we can find an open neighbourhood $W_i \ni w_i$ with $W_i \subset f^k(V_i)$. In particular, $\beta^{i,\epsilon}(t) \in V_i$ for all t sufficiently close to $i/N + \epsilon$, and $\beta^{i,\epsilon}(t) \in V_{i+1}$ for all t sufficiently close to $(i+1)/N + \epsilon$. Thus, by continuity of f, as $\epsilon \to 0$, each of the curves in $\{\beta^{i,\epsilon}\}_i$ converges to a curve with endpoints v_i and v_{i+1}, that we denote by β_i^k. Thus, we can construct the curve

$$\beta_k := \{v_0\} \cdot \beta_0^k \cdot \{v_1\} \cdot \cdots \cdot \beta_N^k \cdot \{v_{N+1}\},$$

which satisfies $\beta_k \in [\gamma]_k$ and $f^k(\beta_k) = \beta$, as required. □

The second goal of this section is to prove Corollary 3.27. This result asserts that given a function f and a domain U in a hyperbolic orbifold, if certain technical conditions are satisfied, then there is a positive constant μ such that for any curve $\gamma \subset U$, there exists a curve in $[\gamma]_0$ of orbifold length less than μ. In the next auxiliary proposition we construct curves in any desired post-0-homotopy class of arbitrarily small orbifold length for orbifolds with a unique ramified point:

Proposition 3.25 (Short post-0-homotopy curves around a ramified point)**.** *Given $\epsilon > 0$ and $d \in \mathbb{N}_{\geq 1}$, define the hyperbolic orbifold $\mathcal{O} := (\mathbb{D}_\epsilon, \nu_d)$ with $\nu_d(0) = d$ and $\nu_d \equiv 1$ elsewhere, and let $\rho_{\mathcal{O}}(z)dz$ be its orbifold metric. Let f be an entire function such that $P(f) \cap \overline{\mathbb{D}}_\epsilon = \{0\}$. Then, for all $\epsilon' < \epsilon$ small enough, $\ell_{\mathcal{O}}(\partial \mathbb{D}_{\epsilon'}) < \epsilon/6$. Moreover, for any curve $\gamma \subset \overline{\mathbb{D}}_{\epsilon'}$, there exists $\tilde{\gamma} \in [\gamma]_0$ satisfying $\ell_{\mathcal{O}}(\tilde{\gamma}) < \epsilon/6$.*

Remark. The function f does not play any role in the proof of the proposition, and its role in the statement is to fix post-0-homotopy classes. Note that for any function f as in the statement and curve $\gamma \subset \mathbb{D}^*_\epsilon$, $[\gamma]_0$ equals the homotopy class of γ (in the usual sense) in the punctured disc $\mathbb{D}^*_\epsilon$.

Proof of Proposition 3.25. Let $\widetilde{\mathcal{O}}$ be the orbifold with underlying surface $\mathbb{D}$ and with 0 as its only ramified point, of degree d. We computed in (3.2.9) an explicit formula for the density of its orbifold metric, namely, for each $u \in \mathbb{D}^*$, it holds $\rho_{\widetilde{\mathcal{O}}}(u)du = 2\left(d|u|^{\frac{d-1}{d}}(1-|u|^{\frac{2}{d}})\right)^{-1}$. If λ_ϵ is the function that factors by ϵ^{-1}, that is, $\lambda_\epsilon(z) := \epsilon^{-1}z$, then $\lambda_\epsilon \colon \mathcal{O} \to \widetilde{\mathcal{O}}$ is an orbifold covering map. Hence, see Observation 2.3, for each $z = \epsilon u$, the density of the orbifold metric of $\mathcal{O}$ is

$$\rho_{\mathcal{O}}(z) = \epsilon^{-1}\rho_{\widetilde{\mathcal{O}}}(u/\epsilon) = 2\left(\epsilon^{1/d}|z|^{\frac{d-1}{d}}\left(1-\epsilon^{-2/d}|z|^{\frac{2}{d}}\right)\right)^{-1}.$$

Observe that for any $\epsilon_1 < \epsilon$, the function $\rho_{\mathcal{O}}$ is constant when restricted to $\partial\mathbb{D}_{\epsilon_1}$, and $\ell_{\mathcal{O}}(\partial\mathbb{D}_{\epsilon_1}) = 2\pi\epsilon_1\rho_{\mathcal{O}}(\epsilon_1)$, as a function of ϵ_1, is strictly decreasing and converging to 0 whenever $\epsilon_1 \to 0$. Thus, the first part of the statement follows.

In order to prove the second part of the statement, note that for any $z \in \mathbb{D}_\epsilon$, the $\mathcal{O}$-length of the radial line joining 0 to z, a segment that we denote by $[0, z]$, is at most $|z|\rho_{\mathcal{O}}(z)$. By a *radial line* we mean any subcurve of a straight line in $\overline{\mathbb{D}}$ joining the origin to $\partial\mathbb{D}$. Thus, the $\mathcal{O}$-length of the segment $[0, z]$ also converges to 0 as $|z| \to 0$. Hence, we can fix any $\epsilon' < \epsilon$ such that

$$\ell_{\mathcal{O}}([0, \epsilon']) < \epsilon/18. \tag{3.5.3}$$

Let $\gamma \subset \mathbb{D}_{\epsilon'}$ with endpoints p and q. If γ contains the point 0, then $\gamma \in H_p^q(\{0\})$ and the concatenation of the radial lines joining p and q to 0, that is, $\tilde{\gamma} := [p, 0] \cdot [0, q]$, satisfies $\tilde{\gamma} \in [\gamma]_0$ and by (3.5.3), $\ell_{\mathcal{O}}(\tilde{\gamma}) < \epsilon/9 < \epsilon/6$. Otherwise, $\gamma \subset H_p^q(\emptyset)$. Thus, the curves in $[\gamma]_0$ are exactly those homotopic to γ (in the usual sense) in $\mathbb{D}_{\epsilon'} \setminus \{0\}$ with fixed endpoints. Note that roughly speaking, the homotopy class of such a curve is determined by the number n of times that the curve "loops" around 0 following an orientation. Hence, we are aiming to construct a representative of any such class with a bound on its orbifold length, namely $\epsilon/6$. In a rough sense, for each $n \geq 0$, we define a representative γ_n^+ as follows: we start at the point p and follow the radial line towards the origin until we meet a circle centred at the origin of some radius ϵ_n small enough. Then we follow anticlockwise an arc of this circle until meeting the point on the radial line from 0 to q. Then we follow the circle of radius ϵ_n anticlockwise n times. Finally, we follow the radial line to q. Similarly, we define a curve γ_n^- starting at q and following the circle of radius ϵ_n clockwise n times.

More formally, for each natural $n \geq 0$, by the observations made at the beginning of the proof, we can choose $\epsilon_n < \epsilon'$ such that

$$\ell_{\mathcal{O}}(\partial\mathbb{D}_{\epsilon_n}) < \frac{\epsilon}{18(n+1)}. \tag{3.5.4}$$

Define $[p, x(n)]$ and $[y(n), q]$ as the restriction of the radial lines from p to 0 and 0 to q with respective endpoints $\{x(n), y(n)\} \subset \partial\mathbb{D}_{\epsilon_n}$. Let α_n^+ and β_n^- be the arcs in $\partial\mathbb{D}_{\epsilon_n}$ that connect $x(n)$ to $y(n)$ in positive and negative orientation respectively. See Figure 5. Let $\partial\mathbb{D}_{\epsilon_n}^+$ and $\partial\mathbb{D}_{\epsilon_n}^-$ be the loops starting at $y(n)$ positively and negatively oriented respectively. We define the curves γ_n^+ and γ_n^-

as the concatenations

$$\gamma_n^+ := [p, x(n)] \cdot \alpha^+ \cdot \underbrace{\partial\mathbb{D}^+_{\epsilon_n} \cdot \cdots \cdot \partial\mathbb{D}^+_{\epsilon_n}}_{n \text{ times}} \cdot [y(n), q] \quad \text{and}$$

$$\gamma_n^- := [p, x(n)] \cdot \beta^- \cdot \underbrace{\partial\mathbb{D}^-_{\epsilon_n} \cdot \cdots \cdot \partial\mathbb{D}^-_{\epsilon_n}}_{n \text{ times}} \cdot [y(n), q].$$

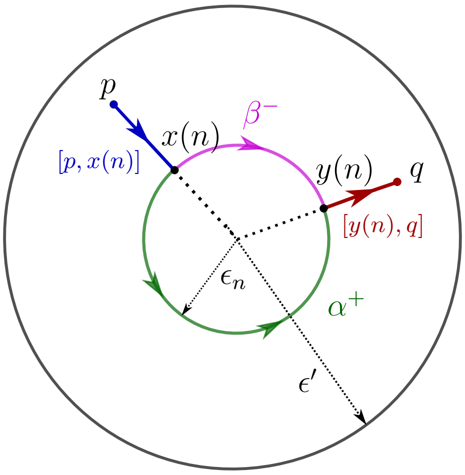

Figure 5: Construction of representatives γ_n^- and γ_n^+ for each post-0-homotopy class of curves in Proposition 3.25 as a concatenation of oriented curves.

By the choices of ϵ' and ϵ_n in (3.5.3) and (3.5.4), $\max\{\ell_{\mathcal{O}}(\gamma_n^-), \ell_{\mathcal{O}}(\gamma_n^+)\} < \epsilon/6$, and thus, for each homotopy class of curves in $\mathbb{D}_{\epsilon'} \setminus \{0\}$, we have constructed an element on it with the desired length. The statement now follows. □

Given an entire function f, in order to construct in Corollary 3.27 curves of any post-0-homotopy class with uniformly bounded orbifold length in a compact set U, we will assume that there are dynamic rays landing at every point in $P(f) \cap U$. The reason for this is that we will use those dynamic rays as a boundary that other dynamic rays cannot cross more than once. Then, Corollary 3.27 will be a consequence of the more general Theorem 3.26, that shows that we can find curves in any desired post-0-homotopy class in any simply connected domain C for which $P(f) \cup \overline{C} \subset \partial C$ and $\#(P(f) \cap \partial C)$ is finite.

If $\pi : \mathbb{D} \to C$ is the Riemann map for some simply connected domain C, whenever ∂C is locally connected, by Carathéodory-Torhorst's theorem[6], π extends continuously to a surjective map $\pi : \overline{\mathbb{D}} \to \overline{C}$, that we call the *extended Riemann map*. Note that in that case, there might exist curves $\gamma \subset \partial C$ for which there is not a curve $\beta \subset \pi^{-1}(\gamma)$ satisfying $\pi(\beta) = \gamma$. For example, let C be a disc

[6]See footnote 2.

D minus a cross "+" that intersects ∂D at a single point. Then, the horizontal segment of the cross would be an example of such a curve in ∂C. We will exclude those "pathological cases" in our result:

Theorem 3.26 (Curves in post-0-homotopy classes with uniformly bounded lengths)**.** *Let f be an entire map and let $\mathcal{O} = (S, \nu)$ be a hyperbolic orbifold with $S \subset \mathbb{C}$. Let $C \subset S \setminus P(f)$ be a simply connected domain such that $C \Subset S$, ∂C is locally connected and $\partial C \cap P(f)$ is finite. Let $\pi : \overline{\mathbb{D}} \to \overline{C}$ be the extended Riemann map. Then, there exists a constant $\eta > 0$ with the following property. Let γ be any curve such that either int(γ) $\subset C$, or $\gamma \subset \partial C$ and there exists a curve $\beta \subset \pi^{-1}(\gamma)$ satisfying $\pi(\beta) = \gamma$. Then there exists a curve $\tilde{\gamma} \in [\gamma]_0$ such that $\ell_{\mathcal{O}}(\tilde{\gamma}) \leq \eta$.*

Remark. We note that the constant η in the statement of the theorem depends on the geometry of C, and in particular on its boundary. Hence, we are not claiming the constant to be uniform for all simply connected domains C satisfying the hypotheses above.

Proof of Theorem 3.26. For any two points $z, w \in \overline{\mathbb{D}}$, we denote by $[z, w]$ the straight segment joining them. We start by finding for each curve of the form $\pi([z, w]) \subset \overline{C}$, a curve in its post-0-homotopy class of (uniformly) bounded $\mathcal{O}$-length. With that aim, let $L : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{R}_{\geq 0}$ be given by

$$L(z, w) := \inf_{\beta \in [\pi([z,w])]_0} \ell_{\mathcal{O}}(\beta). \tag{3.5.5}$$

We claim that L achieves a maximum value μ in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. To prove this, firstly we note that since any geodesic in $\mathcal{O}$ joining two points in $\overline{C}$ has by definition finite $\mathcal{O}$-length, $L(z, w) < \infty$ for all $(z, w) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Then, we show in the next claim that L is upper semicontinuous, and the existence of the maximum follows from the combination of these two facts.

Claim. The function L is upper semicontinuous.

Proof of claim. Let $(z, w) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ and $\epsilon > 0$ be arbitrary but fixed. We want to show that there exists a neighbourhood $U(z) \times U(w)$ of (z, w) such that for every $(\tilde{z}, \tilde{w}) \in U(z) \times U(w)$, $L(\tilde{z}, \tilde{w}) < L(z, w) + \epsilon$. Since by assumption $\#(\overline{C} \cap P(f)) < \infty$, $d := \max_{z \in \overline{C} \cap P(f)} \nu(z) < \infty$. Let us choose $\epsilon' < \epsilon$ small enough so that the estimates provided by Proposition 3.25 with the parameters ϵ and $d!$ hold. Moreover, since the set of ramified points of $\mathcal{O}$ is discrete, we can choose $\epsilon' < \epsilon/3$ such that $\mathbb{D}_{\epsilon'}(\pi(z)) \cup \mathbb{D}_{\epsilon'}(\pi(w)) \subset \mathcal{O}$ and the only possible ramified points in $\mathbb{D}_{\epsilon'}(\pi(z)) \cup \mathbb{D}_{\epsilon'}(\pi(w))$ are $\pi(z)$ and $\pi(w)$. We also choose ϵ' small enough such that $\mathbb{D}_{\epsilon'}(\pi(z)) \cap \mathbb{D}_{\epsilon'}(\pi(w)) = \emptyset$. For the rest of the proof of the claim, we assume that $\pi(z)$ and $\pi(w)$ are ramified points of $\mathcal{O}$ of degree $d!$, since by Corollary 2.7, any estimates on lengths of curves obtained in this setting also hold for the original ramification values of $\pi(z)$ and $\pi(w)$, that lie between 1 and d.

By continuity of π, we can find connected neighbourhoods $U(z) \ni z, U(w) \ni w$ relatively open in $\mathbb{D}$ and satisfying the following properties:

- $\pi(U(z)) \cup \pi(U(w)) \subset (\mathbb{D}_{\epsilon'}(\pi(z)) \cup \mathbb{D}_{\epsilon'}(\pi(w))) \cap \overline{C}$.
- For any $(\tilde{z}, \tilde{w}) \in U(z) \times U(w)$, there exists at least one arc contained in the curve $\pi^{-1}(\partial\mathbb{D}_{\epsilon'}(\pi(z)))$, that we denote by $\xi^{\tilde{z}}$, that joins the first point of intersection of $[z, w]$ with $\pi^{-1}(\partial\mathbb{D}_{\epsilon'}(\pi(z)))$, with the first point of intersection of $[\tilde{z}, \tilde{w}]$ with $\pi^{-1}(\partial\mathbb{D}_{\epsilon'}(\pi(z)))$. Similarly, there exists an arc $\xi^{\tilde{w}}$ in $\pi^{-1}(\partial\mathbb{D}_{\epsilon'}(\pi(w)))$ with analogous properties. See Figure 6.

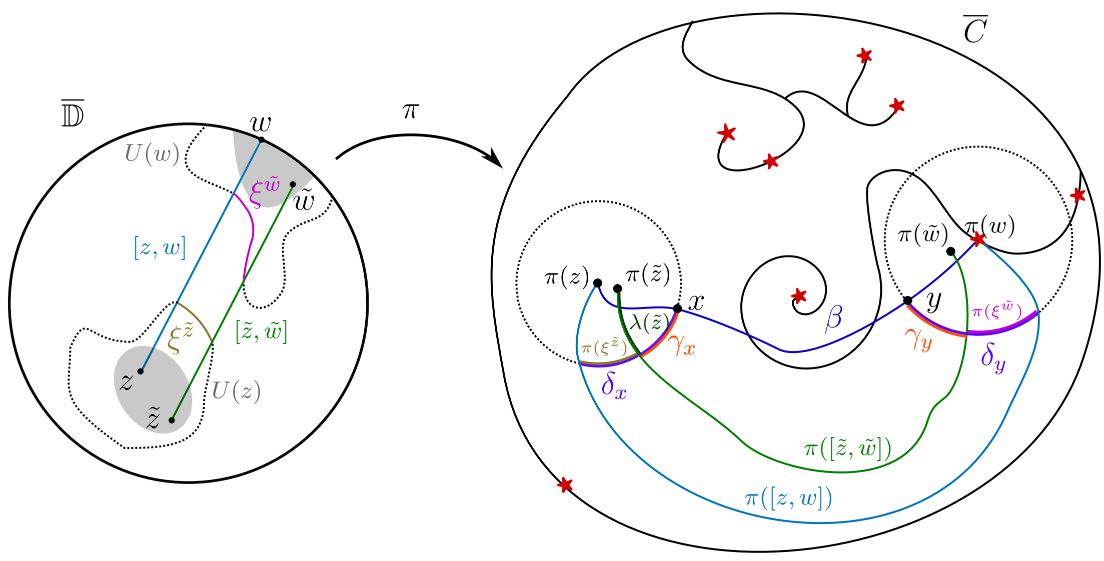

Figure 6: Proof of upper semicontinuity of the function L. Points in $P(f)$ are represented by red stars.

In particular, $\pi(\xi^{\tilde{z}})$ and $\pi(\xi^{\tilde{w}})$ are arcs in $\partial\mathbb{D}_{\epsilon'}(\pi(z)) \cap C$ and $\partial\mathbb{D}_{\epsilon'}(\pi(w)) \cap C$ joining $\pi([\tilde{z}, \tilde{w}])$ and $\pi([z, w])$. Let $\lambda(\tilde{z})$ be the restriction of $\pi([\tilde{z}, \tilde{w}])$ between $\pi(\tilde{z})$ and the endpoint of $\pi(\xi^{\tilde{z}})$ that also belongs to $\pi([\tilde{z}, \tilde{w}])$. In particular, $\lambda(\tilde{z})$ belongs to $\overline{\mathbb{D}}_{\epsilon'}(\pi(z))$, and thus, by Proposition 3.25 there exists $\tilde{\lambda}(\tilde{z}) \in [\lambda(\tilde{z})]_0$ satisfying $\ell_{\mathcal{O}}(\tilde{\lambda}(\tilde{z})) < \epsilon/6$. Analogously, if $\lambda(\tilde{w})$ is the restriction of $\pi([\tilde{z}, \tilde{w}])$ between $\pi(\tilde{w})$ and the endpoint of $\pi(\xi^{\tilde{w}})$ that also belongs to $\pi([\tilde{z}, \tilde{w}])$, then there exists $\tilde{\lambda}(\tilde{w}) \in [\lambda(\tilde{w})]_0$ such that $\ell_{\mathcal{O}}(\tilde{\lambda}(\tilde{w})) < \epsilon/6$.

Consider the subcurves

$$\lceil \pi([\tilde{z},\tilde{w}])\rceil := \pi([\tilde{z},\tilde{w}]) \setminus (\mathbb{D}_{\epsilon'}(\pi(z)) \cup \mathbb{D}_{\epsilon'}(\pi(w))) \quad \text{and}$$
$$\lceil \pi([z,w])\rceil := \pi([z,w]) \setminus (\mathbb{D}_{\epsilon'}(\pi(z)) \cup \mathbb{D}_{\epsilon'}(\pi(w))).$$

In particular, for each of the just defined restrictions, one of their endpoints is an endpoint of $\pi(\xi^{\tilde{z}})$, and the other one is an endpoint of $\pi(\xi^{\tilde{w}})$. Since all curves with fixed endpoints totally contained in a simply connected domain are homotopic, see [Hat02, Proposition 1.6], any two curves totally contained in C are homotopic in $\mathbb{C} \setminus P(f)$, and thus the concatenation

$$\pi(\xi^{\tilde{z}}) \cdot \lceil \pi([z,w])\rceil \cdot \pi(\xi^{\tilde{w}}) \quad \text{is post-0-homotopic to} \quad \lceil \pi([\tilde{z},\tilde{w}])\rceil. \tag{3.5.6}$$

Let us choose[7] any curve $\beta \in [\pi([z,w])]_0$ such that $\ell_{\mathcal{O}}(\beta) < L(z,w) + \epsilon/3$ and $\beta \cap \lceil \pi([z,w])\rceil = \emptyset$. Let x be the first point in $\beta \cap \partial\mathbb{D}_{\epsilon'}(\pi(z))$ and y be last point in $\beta \cap \partial\mathbb{D}_{\epsilon'}(\pi(w))$ with respect to a parametrization of β from $\pi(z)$ to $\pi(w)$, and let $\lceil\beta\rceil$ be the restriction of β between those points. See Figure 6. Let us choose a pair of arcs $\delta_x \subset \partial\mathbb{D}_{\epsilon'}(\pi(z))$ and $\delta_y \subset \partial\mathbb{D}_{\epsilon'}(\pi(w))$ connecting respectively x and y to the single points in the intersections $\pi(\xi^{\tilde{z}}) \cap \lceil \pi([\tilde{z},\tilde{w}])\rceil$ and $\pi(\xi^{\tilde{w}}) \cap \lceil \pi([\tilde{z},\tilde{w}])\rceil$, in such a way that the region that those arcs together with $\lceil\beta\rceil$ and $\lceil \pi([\tilde{z},\tilde{w}])\rceil$ enclose does not contain $\pi(z)$ nor $\pi(w)$. This region always exists by the choice of $\beta \cap \lceil \pi([z,w])\rceil = \emptyset$. Since by assumption $\beta \in [\pi([z,w])]_0$, by construction, the concatenation

$$\delta_x \cdot \lceil \pi([z,w])\rceil \cdot \delta_y \quad \text{is post-0-homotopic to} \quad \lceil\beta\rceil.$$

Consequently, if $\gamma_x \subset (\pi(\xi^{\tilde{z}}) \cup \delta_x)$ and $\gamma_y \subset (\pi(\xi^{\tilde{w}}) \cup \delta_y)$ are the curves joining the endpoints of $\lceil \pi([\tilde{z},\tilde{w}])\rceil$ and $\lceil\beta\rceil$, then, using (3.5.6),

$$\gamma_2 := \gamma_x \cdot \lceil\beta\rceil \cdot \gamma_y \quad \text{is post-0-homotopic to} \quad \lceil \pi([\tilde{z},\tilde{w}])\rceil.$$

By construction and using Proposition 3.25,

$$\ell_{\mathcal{O}}(\gamma_2) \leq \ell_{\mathcal{O}}(\partial\mathbb{D}_{\epsilon'}(\pi(z))) + \ell_{\mathcal{O}}(\beta) + \ell_{\mathcal{O}}(\partial\mathbb{D}_{\epsilon'}(\pi(w))) < \ell_{\mathcal{O}}(\beta) + \epsilon/3.$$

Finally, the concatenation

$$\gamma := \tilde{\lambda}(\tilde{z}) \cdot \gamma_2 \cdot \tilde{\lambda}(\tilde{w}) \quad \text{is post-0-homotopic to} \quad \pi([\tilde{z},\tilde{w}])$$

[7]We believe that the infimum in (3.5.5) is in fact a minimum, that is, we can choose a curve β that is an *orbifold geodesic* in the corresponding post-0-homotopy class with minimum length. Nonetheless, a reference has not been located and its existence is not required for our purposes.

and $L(\tilde{z}, \tilde{w}) \leq \ell_{\mathcal{O}}(\gamma) < L(z, w) + \epsilon$. $\triangle$

If μ is the maximum value that L attains in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, then for every point $(z, w) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$, we can find a curve $\beta \in [\pi([z, w])]_0$ such that $\ell_{\mathcal{O}}(\beta) < 2\mu$. Let γ be a curve such as in the statement. We start by considering both of the cases when $\text{int}(\gamma) \subset C$, and when $\gamma \subset \partial C$ and in addition $\text{int}(\gamma) \cap P(f) = \emptyset$. Let p and q be the endpoints of γ and let $z, w \in \pi^{-1}(\{p, q\})$ be the endpoints of a curve in $\overline{\mathbb{D}}$ that is mapped univalently to γ under π . Note that such curve always exists: when $\text{int}(\gamma) \subset C$, it is the curve that contains the unique preimage $\pi^{-1}(\text{int}(\gamma))$, and when $\gamma \subset \partial C$, there are two such curves, that in particular share one of their endpoints. In both cases, $\pi([z, w])$ together with γ enclose a simply connected domain contained in $(\mathbb{C} \setminus P(f)) \cup \{p, q\}$. Thus, if we consider the set $W := \gamma \cap P(f)$, which might be either empty or contain one of both of the endpoints $\{p, q\}$ of γ, we have, using again [Hat02, Proposition 1.6], that $\pi([z, w])$ and γ are post-0-homotopic, and in particular, $\pi([z, w]) \in [\gamma]_0$. Thus, there exists $\tilde{\gamma} \in [\pi([z, w])]_0 = [\gamma]_0$ such that $\ell_{\mathcal{O}}(\tilde{\gamma}) \leq 2\mu$.

We are left to consider the case when $\gamma \subset \partial C$ and there is a curve $\beta \subset \pi^{-1}(\gamma)$ satisfying $\pi(\beta) = \gamma$. Let p and q be the endpoints of γ and suppose that $\gamma \in H_p^q(W)$ for $W := P(f) \cap \gamma = \{w_1, \dots, w_N\}$. Let us parametrize γ so that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma(\frac{j}{N+1}) = w_j$ for all $1 \leq j \leq N$. In particular, following (3.5.2), we can express γ as a concatenation $\gamma = \gamma_0 \cdot \gamma_2 \cdots \gamma_N$, where $\gamma_i := \gamma|_{[i/N, (i+1)/N]}$. Note that for each $0 \leq i \leq N$, γ_i satisfies the hypotheses of the case considered above, that is, $\gamma_i \subset \partial C$ and $\text{int}(\gamma_i) \cap P(f) = \emptyset$. Thus, for each i there exists a curve $\tilde{\gamma}_i \in [\gamma_i]_0$ with $\ell_{\mathcal{O}}(\tilde{\gamma}_i) \leq 2\mu$. Then, the concatenation $\tilde{\gamma} = \tilde{\gamma}_0 \cdot \tilde{\gamma}_2 \cdots \tilde{\gamma}_N$ satisfies $\tilde{\gamma} \in [\gamma]_0$ and $\ell_{\mathcal{O}}(\tilde{\gamma}) \leq 2\mu |P(f)| =: \nu$. Letting $\eta := \max\{\nu, 2\mu\}$ the theorem follows. $\square$

Corollary 3.27 (Pieces of rays with uniformly bounded length)**.** *Let $f \in \mathcal{B}$, let $\mathcal{O} = (S, \nu)$ be a hyperbolic orbifold with $S \subset \mathbb{C}$ and let $U \Subset S$ be a simply connected domain with locally connected boundary. Assume that $P(f) \cap \overline{U} \subset J(f)$, $\#(P(f) \cap \overline{U})$ is finite and there exists a dynamic ray landing at each point in $P(f) \cap U$. Then, there exists a constant $L_U \geq 0$, depending only on U, such that for any (connected) piece of ray tail $\xi \subset U$, there exists $\delta \in [\xi]_0$ with $\ell_{\mathcal{O}}(\delta) \leq L_U$.*

Proof. Let $P(f) \cap U =: \{p_1, \dots, p_N\}$ for some $N < \infty$. We start by defining a set $X \supset (P(f) \cap U)$ using pieces of dynamic rays. By assumption, for each $1 \leq i \leq N$, there exists at least one dynamic ray landing at each $p_i \in P(f) \cap U$. We choose any such ray and let $\varGamma_i$ be its parametrization including its landing point. Then, denote by γ_i the unique connected component of $\varGamma_i \cap U$ that contains its landing point p_i. We construct the set X inductively: define $X_1 := \gamma_1 \cup \partial U$. For each $2 \leq i \leq N$, let X_i be the union of X_{i-1} with the connected component of $\gamma_i \setminus X_{i-1}$ containing p_i. By construction, $X := X_N$ is a collection of $\tilde{N} \leq N$ connected components, each of them consisting of a concatenation of finitely many pieces of ray tails, and so that $U \setminus X$ is simply connected. That is, the set X can be written as $X = \bigcup_{k=1}^{\tilde{N}} T_k \cup \partial U$, where each T_k is topologically a *tree* with finitely many edges $\{e_1^k, \dots, e_{m(k)}^k\}$. We denote $M := \max_{k \leq \tilde{N}} m(k)$.

Let $C := U \setminus X$ and note that by construction, C is a simply connected domain such that $C \cap P(f) = \emptyset$, and moreover, since $\partial C = X \cup \partial U$, ∂C is locally connected. Hence, the set C satisfies the hypotheses in Theorem 3.26. We claim that if ξ is a piece of ray tail in $U = \overline{C} \setminus \partial U$, then ξ is of one of the following three types:

Type 1. $\text{int}(\xi) \subset C$.

Type 2. $\xi \subset X$ and ξ is a concatenation of at most M curves $\{\alpha_i\}_{i \leq M}$, so that for each α_i there exists a curve $\beta_i \in \partial\mathbb{D}$ such that $\pi(\beta_i) = \alpha_i$, where $\pi\colon \overline{\mathbb{D}} \to \overline{C}$ is the extended Riemann map.

Type 3. ξ is a concatenation of at most $2\tilde{N} + 1$ curves of types 1 and 2.

Indeed, if $\xi \subset \partial C$, by assumption $\xi \subset T_k$ for some k, and so ξ is contained in a concatenation of some of the edges $\{e_1^k, \dots, e_{m(k)}^k\}$ of T_k. For each i, $\pi^{-1}(e_i^k)$ is either an arc that maps 2-to-1 to e_i^k, or consists of two different arcs, each of them mapping 1-to-1 to e_i^k. Thus, ξ is of Type 2. Let us now analyse the case when $\xi \subset \overline{C} \setminus \partial U$ is a piece of dynamic ray which is not of type 1 nor 2. Then $\xi \cap T_k \neq \emptyset$ for some k. Since T_k is a union of pieces of dynamic rays, all of its points but maybe some endpoints escape uniformly to infinity. Hence, $\xi \cap T_k$ must be connected, since otherwise, $T_k \cup \xi$ would enclose a domain that escapes uniformly to infinity, contradicting that $I(f)$ has empty interior as $f \in \mathcal{B}$, see Theorem 2.17. This means that $\xi \cap X$ is a collection of at most $\tilde{N}$ curves, preceded and/or followed by subcurves of ξ with interior in C. Thus, ξ is of Type 3. Hence, by Theorem 3.26, there exists a constant η such that if ξ is a piece of dynamic ray in U, then there exists a curve $\tilde{\xi} \in [\xi]_0$ such that $\ell_{\mathcal{O}}(\tilde{\xi}) \leq \max\{M, 2\tilde{N}+1\}\eta =: \mu$. $\square$

Chapter 4

Splitting hairs with functions in $\mathcal{CB}$

4.1 Signed addresses for criniferous functions

Recall that in Section 2.4 we defined external addresses for each $f \in \mathcal{B}$, and in particular for some but not all points in $I(f)$, see Observation 2.22. In this section, given some additional assumptions on f, we introduce a new form of address generalizing Definition 2.21 and so that all points in $I(f)$ have (at least) one of these new addresses, that we call *signed addresses*. More specifically, our aim is to define signed addresses for criniferous functions in class $\mathcal{B}$ that do not have asymptotic values in their Julia sets. In particular, we consider functions that might contain escaping critical values. Hence, in a very rough sense, any sensible analogue of their Julia constituents, see (2.4.2), that is, satisfying properties (2.4.3) and (2.4.10), would have to consider "bifurcations" or "splitting" at critical points. To illustrate this, we develop further the analysis for the map $f = \cosh$ performed in the introduction.

Example 4.1 (Signed addresses for cosh)**.** Recall that for the map $f = \cosh$, $S(f) = \operatorname{CV}(f) = \{-1, 1\}$, and so we can define tracts and fundamental domains for f using a disc $D \supset \{-1, 1\}$ and letting δ be the piece of positive imaginary axis connecting ∂D to infinity, see Definition 2.18. For this choice of D and δ, each fundamental domain of f is contained in one of the horizontal half-strips

$$\begin{aligned} S_{n_L} &:= \{z : \operatorname{Re} z < 0, \operatorname{Im} z \in ((n-1/2)\pi, (n+3/2)\pi)\} \quad \text{or} \\ S_{n_R} &:= \{z : \operatorname{Re} z > 0, \operatorname{Im} z \in ((n-3/2)\pi, (n+1/2)\pi)\}. \end{aligned} \tag{4.1.1}$$

See 5.30 for more details. Thus, if we label the fundamental domains of f the same way as the strip they belong to, it is easy to see that for the external address $\overline{0_R}$, its Julia constituent $J_{\overline{0_R}} \subset \mathbb{R}^+$. Moreover, $J_{\overline{0_R}}$ and in fact $\mathbb{R}^+$ are ray tails.

We shall *extend* the curve $J_{\overline{0_R}}$ in different ways so that the extensions are still ray tails. By (2.4.3), $f(J_{\overline{0_R}}) \subset J_{\sigma(\overline{0_R})} = J_{\overline{0_R}}$ and thus, there exists a preimage of

$J_{\overline{0_R}}$ that contains $J_{\overline{0_R}}$. Let us denote that preimage by $J^0_{\overline{0_R}}$. If $J^0_{\overline{0_R}} \cap \operatorname{Crit}(f) = \emptyset$, then $J^0_{\overline{0_R}}$ is by definition a ray tail. In that case, we denote by $J^1_{\overline{0_R}}$ the preimage of $J^0_{\overline{0_R}}$ that contains $J^0_{\overline{0_R}}$. We can iterate this process until for some $n \in \mathbb{N}$, a preimage β of $J^n_{\overline{0_R}}$ contains the critical point 0. As explained in the introduction, β is no longer a ray tail, but instead, $\beta \setminus [0, -i\pi/2]$ and $\beta \setminus [0, i\pi/2]$, where $[0, \pm i\pi/2]$ are vertical segments in the imaginary axis, are ray tails. Thus, a choice has to be made on how to define $J^{n+1}_{\overline{0_R}}$. However, if we extend in the same fashion other Julia constituents $J_{\underline{s}_i}$ for addresses $\underline{s}_i$ "sufficiently close" to $\overline{0_R}$, a more careful analysis would show that whenever $\underline{s}_i \to \overline{0_R}$ "from above" (see 2.31), $J^{n+1}_{\underline{s}_i} \to [0, i\pi/2] \cup \mathbb{R}^+$, and whenever $\underline{s}_i \to \overline{0_R}$ "from below", $J_{\underline{s}_i} \to [0, -i\pi/2] \cup \mathbb{R}^+$. Hence, for an analogue of property (2.4.10) to hold, we would have to extend $J_{\overline{0_R}}$ to include both of those two segments. But then, such extension would not be a ray tail. We resolve this obstacle by considering two copies of $\operatorname{Addr}(f)$ indexed by $\{-, +\}$ and defining two ray tails $J^{n+1}_{(\overline{0_R},+)} := [0, i\pi/2] \cup \mathbb{R}^+$ and $J^{n+1}_{(\overline{0_R},-)} := [0, -i\pi/2] \cup \mathbb{R}^+$. By providing $\operatorname{Addr}(f) \times \{-, +\}$ with the "right" topology, an expression similar to (2.4.10) holds for the elements in $\operatorname{Addr}(f) \times \{-, +\}$, that we call signed addresses.

We now formalize these ideas with more generality:

4.2 (Space of signed addresses)**.** Let $f \in \mathcal{B}$ for which a set $\operatorname{Addr}(f)$ of admissible external addresses has been defined. Let us consider the set

$$\operatorname{Addr}(f)_{\pm} := \operatorname{Addr}(f) \times \{-, +\},$$

that we endow with a topology: let $<_\ell$ be the lexicographical order in $\operatorname{Addr}(f)$ defined in 2.31, and let us give the set $\{-, +\}$ the order $\{-\} \prec \{+\}$. Define the linear order

$$(\underline{s}, *) <_{_A} (\underline{\tau}, \star) \qquad \text{if and only if} \qquad \underline{s} <_\ell \underline{\tau} \quad \text{or} \quad \underline{s} =_\ell \underline{\tau} \text{ and } * \prec \star, \tag{4.1.2}$$

where the symbols "$*, \star$" denote generic elements of $\{-, +\}$. This linear order gives rise to a cyclic order: for $a, x, b \in \operatorname{Addr}(f)_{\pm}$,

$$[a, x, b]_{_A} \quad \text{if and only if} \quad a <_{_A} x <_{_A} b \quad \text{or} \quad x <_{_A} b <_{_A} a \quad \text{or} \quad b <_{_A} a <_{_A} x. \tag{4.1.3}$$

In turn, this cyclic order allows us to define a *cyclic order topology* τ_A in $\operatorname{Addr}(f)_{\pm}$.

Definition 4.3 (Signed external addresses for criniferous functions)**.** Let $f \in \mathcal{B}$ be a criniferous function and let $(\operatorname{Addr}(f)_{\pm}, \tau_A)$ be the corresponding topological space defined according to 4.2. A *signed (external) address* for f is any element of $\operatorname{Addr}(f)_{\pm}$.

For each criniferous function $f \in \mathcal{B}$ such that $J(f) \cap \mathrm{AV}(f) = \emptyset$, we aim to define signed external addresses for all points in $I(f)$ by extending subcurves of Julia constituents in a systematic way, as described in Example 4.1. In order to do so, we start by settling which extensions will be allowed at critical points, and defining *canonical tails* as curves in the escaping set that agree with the criterion established.

4.4 (Extensions at critical points)**.** Let $f \in \mathcal{B}$ and let δ be either a ray tail or a dynamic ray (possibly together with its endpoint) such that $\delta \cap \mathrm{AV}(f) = \emptyset$ and also $\delta \cap \mathrm{CV}(f) \neq \emptyset$. Let β be a connected component of $f^{-1}(\delta)$ such that $\beta \cap \mathrm{Crit}(f) \neq \emptyset$. Then, each critical point $c \in \beta$ is the endpoint of $2\deg(f, c)$ curves in $\beta \setminus \mathrm{Crit}(f)$. We denote the set of all such curves by $\mathcal{L}(c)$ and note that topologically, each of them is a radial line from c. See Figure 7. For each $\alpha \in \mathcal{L}(c)$, let $\alpha^-, \alpha^+ \in \mathcal{L}(c)$ be the successor and predecessor curves of α with respect to the anticlockwise circular order of (topological) radial segments in $\mathcal{L}(c)$. Note that by construction, the concatenations $\alpha^- \cdot \{c\} \cdot \alpha$ and $\alpha^+ \cdot \{c\} \cdot \alpha$ are mapped univalently by f to a subset of δ.

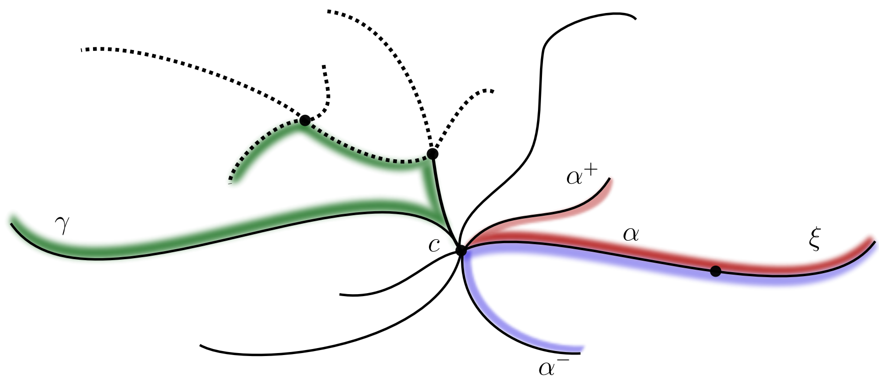

Figure 7: Definition of bristles and canonical tails. In the picture, critical points are represented by black dots and curves in $\mathcal{L}(c)$ with continuous strokes. The curve γ, shown in green, is a canonical tail, and in particular a left-extended curve. The curves α^+ and α^- are the respective right and left bristles of the curve α, that has c as an endpoint.

Definition 4.5 (Canonical tails and rays)**.** Following 4.4, we define:

- The curves α^- and α^+ are the respective *left* and *right bristles* of α.
- For any curve $\xi \subset \beta$ such that $\xi \cap \mathcal{L}(c) = \alpha$ for some $\alpha \in \mathcal{L}(c)$, the concatenations

$$\alpha^- \cdot \{c\} \cdot \xi \quad \text{and} \quad \alpha^+ \cdot \{c\} \cdot \xi$$

  are the respective *left* and *right extensions of* ξ at c.

- Let $\lambda \subset \beta$ be an unbounded simple curve with finite endpoint c_0, and suppose that $\lambda \cap \operatorname{Crit}(f) = \{c_1, \ldots, c_n\}$ for some $n \in \mathbb{N}$, where the points c_i are ordered from smallest to largest potential. For each i, let λ_i be the unbounded curve in $\lambda \setminus \{c_i\}$. If for all $0 \leq i \leq n-1$, λ_i is a right (resp. left) extension of λ_{i+1} at c_{i+1}, then we say that λ is a *right-extended curve* (resp. *left-extended curve*).
- If γ is ray tail (resp. dynamic ray possibly with its endpoint) for which for all $n \geq 0$ such that $\operatorname{Crit}(f) \cap f^n(\gamma) \neq \emptyset$, the curve $f^n(\gamma)$ is either a right-extended or left-extended curve, all of the same type, then we say that γ is a *canonical tail* (resp. *canonical ray*) *of f.*

Remark. If $\gamma \subseteq J^\infty_{\underline{s}}$ is a ray tail (resp. dynamic ray) for some $\underline{s} \in \operatorname{Addr}(f)$, then γ is a canonical tail (resp. ray), since by definition of Julia constituents, $\operatorname{Orb}^+(J^\infty_{\underline{s}}) \cap \operatorname{Crit}(f) = \emptyset$.

In the forthcoming Proposition 4.8, for certain criniferous functions, we will establish a correspondence between canonical tails and signed addresses. We achieve this by extending the curves $J^\infty_{\underline{s}}$ in such a way so that all extensions are canonical curves, and so that all points in $I(f)$ belong to at least one canonical curve. In certain cases, rather than extending directly the curve $J^\infty_{\underline{s}}$, for technical reasons, it is more convenient to extend some unbounded subcurve of $J^\infty_{\underline{s}}$. In fact, this will be the case in Section 4.6. Compare with equation (4.4.11) and Proposition 4.49. The next definition establishes which conditions the mentioned subcurves must fulfil.

Definition 4.6. (Initial configuration of tails) Let $f \in \mathcal{B}$ and suppose that for each $\underline{s} \in \operatorname{Addr}(f)$, there exists a curve $\gamma^0_{\underline{s}} \subset J^\infty_{\underline{s}}$ that is either a ray tail, or a dynamic ray possibly with its endpoint. The set of curves $\{\gamma^0_{\underline{s}}\}_{\underline{s}\in \operatorname{Addr}(f)}$ is a *valid initial configuration* for f if for each $\underline{s} \in \operatorname{Addr}(f)$, $f(\gamma^0_{\underline{s}}) \subset \gamma^0_{\sigma(\underline{s})}$ and

$$I(f) \subset \operatorname{Orb}^-\left(\bigcup_{\underline{s}\in\operatorname{Addr}(f)} \gamma^0_{\underline{s}}\right) =: \mathcal{S}. \tag{4.1.4}$$

Observation 4.7 (Existence of initial configuration equivalent to criniferous)**.** Note that if for a function $f \in \mathcal{B}$ there exists a valid initial configuration, by (4.1.4), f is criniferous. Conversely, if $f \in \mathcal{B}$ is criniferous, by Theorem 2.30, $\{J^\infty_{\underline{s}}\}_{\underline{s}\in\operatorname{Addr}(f)}$ is a valid initial configuration for f. Moreover, note that all curves in a valid initial configuration are canonical and pairwise disjoint, as Julia constituents are.

Proposition 4.8 (Canonical tails for signed addresses)**.** *Let $f \in \mathcal{B}$ be a criniferous function such that $J(f) \cap AV(f) = \emptyset$. Let $\{\gamma^0_{\underline{s}}\}_{\underline{s}\in\operatorname{Addr}(f)}$ be a valid initial* *configuration for f and let $\mathcal{S}$ be the set in (4.1.4). For each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$ let $\gamma^0_{(\underline{s},*)} := \gamma^0_{\underline{s}}$. Then, for every $n \in \mathbb{N}_{\geq 1}$, there exists a canonical tail (resp. ray with possibly its endpoint) $\gamma^n_{(\underline{s},*)}$ such that $\gamma^n_{(\underline{s},*)} \supseteq \gamma^{n-1}_{(\underline{s},*)}$ and f maps $\gamma^n_{(\underline{s},*)}$ bijectively to $\gamma^{n-1}_{(\sigma(\underline{s}),*)}$. In particular, if for each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$ we define the Γ-*curve

$$\Gamma(\underline{s}, *) := \bigcup_{n \geq 0} \gamma^n_{(\underline{s},*)},$$

then $\mathcal{S} = \bigcup_{(\underline{s},*)\in \operatorname{Addr}(f)_\pm} \Gamma(\underline{s}, *)$.

Proof. Without loss of generality and for clarity of exposition, we assume that all curves in the given initial configuration are canonical tails, since our arguments work exactly the same way if any curve is a dynamic ray (possibly with its endpoint). We construct canonical tails inductively on n and simultaneously for all elements in $\operatorname{Addr}(f)_\pm$. Let $n = 1$ and choose any $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$. Since by assumption $f(\gamma^0_{\underline{s}}) \subset \gamma^0_{\sigma(\underline{s})}$, there exists a connected component β of $f^{-1}(\gamma^0_{(\sigma(\underline{s}),*)})$ such that $\gamma^0_{(\underline{s},*)} \subseteq \beta$. Define

$$\gamma^1_{(\underline{s},*)} := \beta,$$

which is a canonical tail since it is a preimage of the canonical tail $\gamma^0_{(\sigma(\underline{s}),*)}$, that by definition does not contain any singular values, and hence $\gamma^1_{(\underline{s},*)} \cap \operatorname{Crit}(f) = \emptyset$. Note that $\gamma^1_{(\underline{s},-)} = \gamma^1_{(\underline{s},+)}$, and so the curve can be regarded both as left-extended or right-extended. However, for the purpose of our inductive argument, we regard $\gamma^1_{(\underline{s},-)}$ as a left-extended curve, and $\gamma^1_{(\underline{s},+)}$ is a right-extended curve.

Suppose that the first statement of the proposition has been proved for some $n \in \mathbb{N}$ and all elements in $\operatorname{Addr}(f)_\pm$. We shall see that it holds for $n + 1$. By the inductive hypothesis, for each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$, both $\gamma^n_{(\underline{s},*)}$ and $\gamma^n_{(\sigma(\underline{s}),*)}$ are well-defined canonical tails, and $f(\gamma^n_{(\underline{s},*)}) = \gamma^{n-1}_{(\sigma(\underline{s}),*)} \subset \gamma^n_{(\sigma(\underline{s}),*)}$. Moreover, $\gamma^n_{(\sigma(\underline{s}),-)}$ must be a left-extended curve and $\gamma^n_{(\sigma(\underline{s}),+)}$ a right-extended curve. Let β be the component of $f^{-1}(\gamma^n_{(\sigma(\underline{s}),*)})$ that contains $\gamma^n_{(\underline{s},*)}$. If $(\beta \setminus \gamma^n_{(\underline{s},*)}) \cap \operatorname{Crit}(f) = \emptyset$, then we denote

$$\gamma^{n+1}_{(\underline{s},*)} := \beta,$$

which is a canonical tail by the same argument as before. Otherwise, $\gamma^n_{(\underline{s},*)}$ must be contained in a unique connected component L_1 of $\beta \setminus (\operatorname{Crit}(f) \setminus \gamma^n_{(\underline{s},*)})$. In particular, $L_1 \setminus \gamma^n_{(\underline{s},*)}$ does not contain any critical points, but can be extended to contain a critical point $c_1 \in \beta$ as finite endpoint. Hence, since by the inductive hypothesis $f|_{\gamma^n_{(\underline{s},*)}}$ maps bijectively to $\gamma^{n-1}_{(\sigma(\underline{s}),*)}$, f maps the curve $\{c_1\} \cup L_1$ univalently to

$\gamma^n_{(\sigma(\underline{s}),*)}$. If $f(\{c_1\}\cdot L_1) = \gamma^n_{(\sigma(\underline{s}),*)}$ we define

$$\gamma^{n+1}_{(\underline{s},*)} := \{c_1\}\cdot L_1$$

and the claim follows. If on the contrary $f(\{c_1\}\cdot L_1) \subsetneq \gamma^n_{(\sigma(\underline{s}),*)}$, then L_1 must be a curve containing a unique element of $\mathcal{L}(c_1)$. Thus, following Definition 4.5, we define L_2 as the respective right or left extension of L_1 at c_1 according to whether $* = +$ or $* = -$. That is, if α^- and α^+ are the respective left and right bristles of L_1 at c_1, then we define

$$L_2 := \begin{cases} \alpha^+\cdot\{c_1\}\cdot L_1 & \text{if } * = +, \qquad \text{or} \\ \alpha^-\cdot\{c_1\}\cdot L_1 & \text{if } * = -. \end{cases}$$

Since β is the preimage of a ray tail, the curve L_2 can be extended to contain an endpoint c_2, and if c_2 is not a critical point, then $f(c_2)$ must be the finite endpoint of $\gamma^n_{(\sigma(\underline{s}),*)}$. If $f(\{c_2\}\cdot L_2) = \gamma^n_{(\sigma(\underline{s}),*)}$, then we define

$$\gamma^{n+1}_{(\underline{s},*)} := \{c_2\}\cdot L_2$$

and the claim follows. This is because by construction, $\{c_2\}\cdot L_2$ is either a right-extended or left-extended curve, depending only on whether $* = +$ or $* = -$, and by the inductive hypothesis, the same applies to the canonical tails $\gamma^n_{(\underline{s},*)}$ and $\gamma^n_{(\sigma(\underline{s}),*)}$, and hence $\gamma^{n+1}_{(\underline{s},*)}$ is a canonical tail. Otherwise, if $f(\{c_2\}\cdot L_2) \neq \gamma^n_{(\sigma(\underline{s}),*)}$, the point c_2 must be a critical point, and we can define L_3 as the right or left extension of L_2 at c_2 following the same criterion as before. Iterating this process we get a collection $\cdots \supset L_{i+1} \supset L_i \supset \cdots$ of right or left extended curves, all of the same type, contained in β. Since $\beta \subset J(f)$ and by assumption $J(f)\cap \mathrm{AV}(f) = \emptyset$, this process must converge. To see this, suppose that the piece of $\gamma^n_{(\sigma(\underline{s}),*)}$ from its finite endpoint p to $f(c_1)$ is parametrized from $[0,1]$. Then, $f^{-1}(\gamma^n_{(\sigma(\underline{s}),*)}([0,1]))\cap\beta$ is bounded, and so the sequence $\{L_i\}_{i\geq 1}$ converges to a canonical tail $L \subset \beta$ such that $f(L) = \gamma^n_{(\sigma(\underline{s}),*)}$. Consequently, by defining

$$\gamma^{n+1}_{(\underline{s},*)} := L,$$

the first part of the proposition follows. The second part is a direct consequence of the construction process together with equation (4.1.4) in Definition 4.6. $\square$

As a consequence of the previous proposition and (4.1.4), for the functions studied in this section, the *strong version of Eremenko's conjecture* [Erë89] holds:

Corollary 4.9 (Strong Eremenko's conjecture holds for certain criniferous functions)**.** *Let $f \in \mathcal{B}$ be a criniferous function such that $J(f) \cap AV(f) = \emptyset$. Then, every $z \in I(f)$ can be connected to infinity by at least one curve γ such that $f^n|_\gamma \to \infty$ uniformly. In particular, γ can be chosen to be a canonical tail.*

Observe that we have shown in Proposition 4.8 for each criniferous function in class $\mathcal{B}$ that does not contain asymptotic values in its Julia set, that its escaping set consists of a collection of $\varGamma$-curves, each of them being a union of nested canonical tails. However, we cannot assert just yet that each of those curves is a dynamic ray, nor that it lands, since it would still be left to show that this union does indeed converge to a curve with a finite endpoint. Nonetheless, each $\varGamma$-curve can still be characterized as a concatenation of pieces of ray tails. In particular, this allows us to study the overlappings occurring within this collection of sets.

Proposition 4.10 (Overlapping of $\varGamma$-curves)**.** *Following Proposition 4.8, for each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$, either $\varGamma(\underline{s}, -) = \varGamma(\underline{s}, +)$ when $Orb^-(\operatorname{Crit}(f)) \cap \varGamma(\underline{s}, *) = \emptyset$, or $\varGamma(\underline{s}, *)$ can be expressed as a concatenation*

$$\varGamma(\underline{s}, *) = \cdots \cdot \{c_{i+1}\} \cdot \gamma_i^{i+1} \cdot \{c_i\} \cdot \cdots \cdot \gamma_0^1 \cdot \{c_0\} \cdot \gamma_{c_0}^{\infty}, \tag{4.1.5}$$

where $\{c_i\}_{i \in I} = Orb^-(\operatorname{Crit}(f)) \cap \varGamma(\underline{s}, *)$, for each $i \geq 1$, if it exists, the curve γ_i^{i+1} is a (bounded) piece of dynamic ray, and $\gamma_{c_0}^\infty$ is a piece of dynamic ray joining c_0 to infinity. In particular, in the latter case, the following properties hold for $\varGamma(\underline{s}, *)$:

(A) $\gamma_{c_0}^\infty \cup \{c_0\} = \varGamma(\underline{s}, -) \cap \varGamma(\underline{s}, +)$ and $\gamma_{c_0}^\infty$ does not belong to any other $\varGamma$-curve.

(B) For each $i \geq 0$, the point c_i belongs to exactly $2\prod_{j=0}^{\infty} \deg(f, f^j(c_i))$ $\varGamma$-curves.

(C) For each $i \geq 0$, $\gamma_i^{i+1} = \varGamma(\underline{s}, *) \cap \varGamma(\underline{\tau}, \star)$, where $\star \neq *$ and $\sigma^j(\underline{\tau}) = \sigma^j(\underline{s})$ for some $j \geq 1$. Moreover, γ_i^{i+1} does not belong to any other $\varGamma$-curve.

Remark. Note that $\mathcal{S} \subset J(f)$ by definition, and since any periodic critical point of f belongs to $F(f)$, for each $i \in I$, the product $2\prod_{j=0}^{\infty} \deg(f, f^j(c_i))$ in (B) is always finite. More specifically, let $n \in \mathbb{N}$ such that the point $c_i \in \gamma^n_{(\underline{s},*)} \setminus \gamma^{n-1}_{(\underline{s},*)}$. Then, by Proposition 4.8 it holds that $f^n(c_i) \subset \gamma^0_{(\sigma^n(\underline{s}),*)}$, and as since by definition $\gamma^0_{(\sigma^n(\underline{s}),*)} \cap \operatorname{Orb}^-(\operatorname{Crit}(f)) = \emptyset$, $\prod_{j=0}^{\infty} \deg(f, f^j(c_i)) = \prod_{j=0}^{n} \deg(f, f^j(c_i)) < \infty$.

Proof of Proposition 4.10. For a fixed $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$, the dichotomy and characterization of $\varGamma(\underline{s}, *)$ in the statement are a direct consequence of its definition in Proposition 4.8; more specifically, suppose that $\operatorname{Orb}^-(\operatorname{Crit}(f)) \cap \varGamma(\underline{s}, *) = \emptyset$. Then, for each $n \geq 0$, there exists an inverse branch of f^n defined in a neighbourhood of $\gamma^0_{(\sigma^n(\underline{s}),*)}$ that maps $\gamma^0_{(\sigma^n(\underline{s}),*)}$ bijectively to the connected component of $f^{-n}(\gamma^0_{(\sigma^n(\underline{s}),*)})$ that contains $\gamma^n_{(\underline{s},-)} = \gamma^n_{(\underline{s},+)}$. If $\mathrm{Orb}^-(\mathrm{Crit}(f)) \cap \Gamma(\underline{s},*) \neq \emptyset$, then since by Proposition 4.8 the curve $\Gamma(\underline{s},*)$ is a union of nested canonical tails, it must be of the form specified in (4.1.5).

For the rest of the proof, for each $n \in \mathbb{N}$ we refer to the elements in

$$L(n) := \left\{\gamma^n_{(\underline{\tau},\star)} : (\underline{\tau},\star) \in \mathrm{Addr}(f)_\pm\right\}$$

as *curves of level* n. We shall use the following observation:

Claim. Suppose that $z \in \gamma \in L(n)$ for some $n \in \mathbb{N}$. Then, $z \in \gamma^m_{(\underline{\tau},\star)}$ for some $(\underline{\tau},\star) \in \mathrm{Addr}(f)_\pm$ and $m > n$ if and only if $z \in \gamma^n_{(\underline{\tau},\star)}$.

In other words, if $z \in \gamma \in L(n)$ for some $n \in \mathbb{N}$, then all the Γ-curves z belongs to are determined by the curves of level n it belongs to.

Proof of claim. If $z \in \gamma^n_{(\underline{\tau},\star)}$ for some $(\underline{\tau},\star) \in \mathrm{Addr}(f)_\pm$, then by Proposition 4.8, $z \in \gamma^m_{(\underline{\tau},\star)}$ for all $m \geq n$. In order to prove the converse, suppose that $z \in \gamma^m_{(\underline{\tau},\star)}$ for some $(\underline{\tau},\star)$ and $m > n$. Then, $f^n(z) \in \gamma^0_{(\sigma^n(\underline{s}),*)} \cup \gamma^{m-n}_{(\sigma^n(\underline{\tau}),\star)}$ by Proposition 4.8. However, since curves of level 0 are pairwise disjoint and are contained only in curves of level n that differ at most in the sign of their addresses, it must occur that $\sigma^n(\underline{s}) = \sigma^n(\underline{\tau})$. This implies that $\gamma^0_{(\sigma^n(\underline{\tau}),\star)} = \gamma^0_{\sigma^n(\underline{s})}$ and $z \in \gamma^n_{(\underline{\tau},\star)}$. △

For the rest of the proof, let us fix an arbitrary $(\underline{s},*) \in \mathrm{Addr}(f)_\pm$. In order to prove (A), we start recalling that all curves in a valid initial configuration are pairwise disjoint, and that by definition, $\gamma^0_{(\underline{s},+)} = \gamma^0_{\underline{s}} = \gamma^0_{(\underline{s},-)}$. Let $n \in \mathbb{N}$ be the smallest number such that $\gamma^n_{(\underline{s},*)} \cap \mathrm{Orb}^-(\mathrm{Crit}(f)) \neq \emptyset$, and let c_0 be the point in that intersection of greatest potential in $\gamma^n_{(\underline{s},*)}$. Then, there exists an inverse branch of f^n defined in a neighbourhood of $f^n(\gamma^\infty_{c_0}) \subset \gamma^0_{(\sigma^n(\underline{s}),*)}$ that maps $f^n(\gamma^\infty_{c_0})$ bijectively to $\gamma^\infty_{c_0}$, and thus, $\gamma^n_{(\underline{s},-)} \cap \gamma^\infty_{c_0} = \gamma^n_{(\underline{s},+)} \cap \gamma^\infty_{c_0}$. In particular, by the claim, $\gamma^\infty_{c_0}$ does not intersect any other canonical tails, and so (A) follows.

We prove items (B) and (C) simultaneously. Note that by the claim, all overlappings of $\gamma^m_{(\underline{s},*)}$ with curves of level $n < m$ occur in $\gamma^n_{(\underline{s},*)} \subset \gamma^m_{(\underline{s},*)}$. By this and the definition of Γ-curves as a union of nested curves of level m, with $m \to \infty$, in order to prove (B) and (C), it suffices to show that for any $n \geq 0$, (B) and (C) hold replacing in the statement of the proposition each $\Gamma(\underline{s},*)$ by its restriction to $\gamma^n_{(\underline{s},*)}$. We proceed to do so by induction on n. For $n = 0$, since curves of level 0 do not contain (preimages of) critical points, the statements hold trivially. Suppose that (B) and (C) hold for some $n \in \mathbb{N}$, and we shall see they hold for $n+1$.

Let us consider the curve $\gamma^{n+1}_{(\underline{s},*)}$. By Proposition 4.8, $f(\gamma^{n+1}_{(\underline{s},*)}) = \gamma^{n}_{(\sigma(\underline{s}),*)}$ and by the inductive hypothesis, the statements hold for both $\gamma^{n}_{(\underline{s},*)}$ and $\gamma^{n}_{(\sigma(\underline{s}),*)}$. In particular, since $\gamma^{n}_{(\underline{s},*)} \subseteq \gamma^{n+1}_{(\underline{s},*)}$, for all γ^{i+1}_{i} and c_i contained in $\gamma^{n}_{(\underline{s},*)}$ for some $i \in I$, by the claim, (B) and (C) hold. Thus, if $\gamma^{n+1}_{(\underline{s},*)} = \gamma^{n}_{(\underline{s},*)}$, we are done. Otherwise, it suffices to prove that they hold for all

$$c_i \in \beta := \gamma^{n+1}_{(\underline{s},*)} \setminus \gamma^{n}_{(\underline{s},*)} \quad \text{and} \quad \gamma^{i+1}_{i} \cap \beta \neq \emptyset \quad \text{for some } i \in I.$$

If $\beta \cap \mathrm{Crit}(f) = \emptyset$, then, arguing as before, there exists a neighbourhood of β that maps injectively to $\gamma^{n}_{(\sigma(\underline{s}),*)}$. In particular, for each curve of level n that contains $f(\beta)$, there exists a unique curve of level $n+1$ that maps to it and also contains β. Then, by the inductive hypothesis applied to curves of level n, (B) and (C) hold for $\beta \cup \gamma^{n}_{(\underline{s},*)} = \gamma^{n+1}_{(\underline{s},*)}$.

Otherwise, let $c \in \beta \cap \mathrm{Crit}(f)$ be the critical point of maximal potential in β. Then, by definition, the map f acts like $z \mapsto z^{\deg(f,c)}$ locally around c. By the inductive hypothesis, $f(c)$ belongs to $N := 2\prod_{j=1}^{\infty} \deg(f, f^j(c))$ curves of level n that by (C) overlap pairwise. Let $\mathcal{L}(c)$ be the set of curves in $\beta \setminus \mathrm{Crit}(f)$ for which c is an endpoint. Then, the cardinal of $\mathcal{L}(c)$ is either $\deg(f,c) \cdot N$, or $2\deg(f,c) \cdot N$, depending on whether $f(c)$ is or not the endpoint of $\gamma^{n}_{(\sigma(\underline{s}),*)}$. We will assume without loss of generality that the second case occurs, since the argument in the first one is a simplified version of the one to follow. Let us subdivide the curves in $\mathcal{L}(c)$ into the respective subsets $\mathcal{L}_b$ and $\mathcal{L}_u$ of curves that are mapped to the bounded or unbounded component of $\gamma^{n}_{(\sigma(\underline{s}),*)} \setminus \{f(c)\}$.

In particular, since $c \in \beta$, $\gamma^{n}_{(\underline{s},*)}$ must belong to a curve in $\mathcal{L}_u$. Moreover, by Proposition 4.8 and the inductive hypothesis, each curve in $\mathcal{L}_u$ contains a pair of curves $\gamma^{n}_{(\underline{\tau},+)}$ and $\gamma^{n}_{(\underline{\alpha},-)}$ for some $\underline{\tau}, \underline{\alpha} \in \mathrm{Addr}(f)$ such that $\sigma(\underline{\tau}) = \underline{s} = \sigma(\underline{\alpha})$. Since f maps each curve in $\mathcal{L}_u$ injectively to $\gamma^{n}_{(\sigma(\underline{s}),*)} \setminus \{f(c)\}$, arguing as before, each curve in $\mathcal{L}_u$ belongs to two curves of level $n+1$ that extend the curves of level n they contain. In addition, since curves of level $n+1$ are canonical tails, for each of them, a curve from $\mathcal{L}_b$ is a bristle. In particular, following the extending criterion from 4.4, each curve in $\mathcal{L}_b$ is both a left bristle for a left-extended curve in $\mathcal{L}_u$, and a right bristle for a right-extended curve in $\mathcal{L}_u$. In particular, one of these right and left-extended curves must belong to $\gamma^{n+1}_{(\underline{s},*)}$, and we have shown that items (B) and (C) hold for the restriction of $\gamma^{n+1}_{(\underline{s},*)}$ to the unbounded component of $\gamma^{n+1}_{(\underline{s},*)} \setminus (\mathrm{Crit}(f) \setminus \{c\})$. If we denote the bounded component by δ, repeating this process iteratively for each critical point in δ, since δ is bounded, the process must converge and the statements follow. $\square$

Definition 4.11 (Signed addresses for escaping points)**.** Under the conditions of Proposition 4.8, for each $z \in \mathcal{S} \supset I(f)$, we say that z has *signed (external) address* $(\underline{s}, *)$ if $z \in \Gamma(\underline{s}, *)$, and we denote by $\operatorname{Addr}(z)_\pm$ the set of all signed addresses of z.

Observation 4.12 (Escaping points have at least two signed addresses)**.** By Proposition 4.10, for each $z \in \mathcal{S}$,

$$\# \operatorname{Addr}(z)_\pm = 2 \prod_{j=0}^{\infty} \deg(f, f^j(z)) < \infty. \tag{4.1.6}$$

Therefore, each point in $\mathcal{S} \supset I(f)$ has at least two signed addresses.

Observation 4.13 (Universality of canonical tails)**.** We note that the concept of *canonical tail* for a criniferous function $f \in \mathcal{B}$ is defined in 4.4 independently of the choice of fundamental domains, and thus external addresses, for f. However, for each choice of $\operatorname{Addr}(f)_\pm$, since any canonical tail, or more generally any ray tail, escapes uniformly to infinity, it must contain some Julia constituent. Then, it follows from Propositions 4.8 and 4.10 that if $\gamma \subset I(f)$ is a canonical tail, then there exists $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$ such that $\gamma = \gamma^n_{(\underline{s},*)}$ for some $n \geq 0$.

Observation 4.14 (Landing of canonical rays implies landing of all rays)**.** For a criniferous function $f \in \mathcal{B}$ such that $J(f) \cap AV(f) = \emptyset$, showing that all its canonical rays land suffices to conclude that all its dynamic rays land. This is because we have shown in Proposition 4.8 that $I(f) \subset \mathcal{S} = \bigcup_{(\underline{s},*) \in \operatorname{Addr}(f)_\pm} \Gamma(\underline{s}, *)$, and so, any dynamic ray γ must belong to $\mathcal{S}$. In particular, if a ray γ is canonical, then by Observation 4.13, it must be contained in $\Gamma(\underline{s}, *)$ for some signed address $(\underline{s}, *)$, and if γ is not canonical, then γ must be a concatenation at (preimages of) critical points of pieces of ray tails, where instead of extending as in Definition 4.5, different choices of bristles are made.

To conclude this section, we provide an overview of how signed addresses will aid us in our goal of proving Theorem 1.8. Firstly, for f satisfying further hypotheses, in §4.2 we construct for each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$ a neighbourhood of $\gamma^0_{(\underline{s},*)}$ that allows us to define for each $n \in \mathbb{N}$ an inverse branch of f^n containing $\gamma^n_{(\underline{s},*)}$ in its image. Moreover, inverse branches will be defined to agree for all addresses in an interval around $(\underline{s}, *)$. In particular, this can be seen as a sort of analogue for signed addresses of the property (2.4.10) for external addresses. Our next step towards the proof of Theorem 1.8 will be to show in Proposition 4.49 that functions in $\mathcal{CB}$ are criniferous, and that it is possible to choose a convenient valid initial configuration of curves to define canonical tails for these maps. Then, in §4.5 we construct a topological model for each $f \in \mathcal{CB}$ by considering two copies of the Julia Cantor bouquet of any disjoint type function on its parameter space. This model reflects the two copies of addresses in $\operatorname{Addr}(f)_\pm$. Finally, in §4.6 we map each component of the model signed with a "+" to a Γ-curve of signed address "$(\cdot, +)$", and each component signed with a "−" to a curve "$\Gamma(\cdot, -)$".

Observation 4.15 (Equivalence of orders)**.** We note that locally, the anticlockwise order of radial segments used in Definition 4.5 agrees with the order in (4.1.3) for the addresses in $\operatorname{Addr}(f)_\pm$, since in (2.4.7) we chose positive orientation. More precisely, we aim to show in §4.2 that by construction, given a converging sequence of points $\{z_k\}_k \subset \mathcal{S}$, for each $k > 0$, there is a choice of $(\underline{s}_k, *_k) \in \operatorname{Addr}(z_k)$ such that

$$\text{if} \quad z_k \to z, \quad \text{then} \quad (\underline{s}_k, *_k) \to (\underline{s}, *) \quad \text{as} \quad k \to \infty \tag{4.1.7}$$

for some $(\underline{s}, *) \in \operatorname{Addr}(z)$. Compare to (2.4.10).

4.2 Fundamental hands and inverse branches

The standing assumptions for this section are the following: we assume that $f \in \mathcal{B}$ is criniferous, $J(f) \cap \operatorname{AV}(f) = \emptyset$ and in addition,

(PF) *The set $P(f) \setminus I(f)$ is bounded and the set $S_I := S(f) \cap I(f)$ is finite.*

In a rough sense, our goal in this section is the following: recall that by Proposition 4.8, given the standing assumptions on f, we can define for each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$ and $n \geq 0$, a canonical tail $\gamma^n_{(\underline{s},*)}$ such that f^n maps $\gamma^n_{(\underline{s},*)}$ bijectively to $\gamma^0_{(\underline{s},*)}$. Using this, we aim to define for each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$ and $n \geq 0$ an inverse branch of f^n on a neighbourhood U of $\gamma^0_{(\underline{s},*)}$ such that the image of this inverse branch contains $\gamma^n_{(\underline{s},*)}$. In addition, U will be defined so that there exists an interval of signed addresses containing $(\underline{s}, *)$ such that the same inverse branch with analogous properties can be taken for all addresses in the interval. See Lemma 4.23. We are able to achieve this result thanks to the consistency on taking always either right or left extensions in the definition of canonical tails, together with the equivalence of orders pointed out in Observation 4.15. In order to define these inverse branches, we introduce the concept of *fundamental hands*, that in a rough sense are n-th preimages of certain simply connected subsets of fundamental domains so that f^n is injective on each of them, see Definition 4.17. Moreover, we point out that for us, the existence of these inverse branches plays an important role in the construction of the desired semiconjugacy of Theorem 1.8, as they allow us to define a sequence of *continuous* functions that are successive better approximations of the map φ from (1.0.2). This will be reflected in Proposition 4.62. See also Figure 12.

Remark. We suggest the reader familiar with [BRG17] to compare the notion of fundamental hands with that of *fundamental tails* for postsingularly bounded functions, introduced in [BRG17, Section 3] in order to define *dreadlocks*. We point out that for the functions we consider in this section, we face the additional challenge of the presence of critical points in their escaping sets, and hence the existence of a sensible generalization of the concept of dreadlocks for our functions is a priori not obvious to us. See §5.2 for further discussion on this topic.

For any function f satisfying the standing hypotheses of this section, we start by defining fundamental domains with respect to a convenient choice of a domain $D \supset S(f)$ and a curve δ connecting $\overline{D}$ to infinity, with the aim of simplifying arguments in future proofs. Recall that by Corollary 4.9, every point in S_I is the endpoint of at least one canonical tail.

Proposition 4.16 (Parameters to define fundamental domains)**.** *Let $f \in \mathcal{B}$ be a criniferous function such that $J(f) \cap AV(f) = \emptyset$ and satisfying (PF). For each $z \in S_I$, let γ_z be a canonical tail with finite endpoint z. Then, there exists a Jordan domain $D \supset S(f) \cup (P(f) \setminus I(f))$ and an arc $\delta \subset \mathbb{C} \setminus \overline{f^{-1}(\mathbb{C} \setminus D)}$ connecting a point of $\overline{D}$ to infinity such that*

$$W_{-1} := \mathbb{C} \setminus (\overline{D} \cup \delta) \tag{4.2.1}$$

has the following property: if for some $n \geq 1$, a connected component τ of $f^{-n}(W_{-1})$ contains a point $z \in S_I$, then $\gamma_z \subset \tau$. Moreover, for any such component τ, $(P(f) \setminus I(f)) \cap \tau = \emptyset$.

Proof. Since $f \in \mathcal{B}$ and satisfies (PF), we can choose a disk $\mathbb{D}_{R_0}$ for some $R_0 > 0$ sufficiently large so that $S(f) \cup (P(f) \setminus I(f)) \subset \mathbb{D}_{R_0}$. Let $\mathcal{T}_{\mathbb{D}_{R_0}} := f^{-1}(\mathbb{C} \setminus \mathbb{D}_{R_0})$ be the corresponding set of tracts. Since each ray tail escapes to infinity uniformly, see Definition 1.5, for each $z \in S_I$ there exists a natural number $N(z) > 0$ such that $f^n(\gamma_z) \subset \mathcal{T}_{\mathbb{D}_{R_0}}$ for all $n \geq N(z)$. Let us consider the set of ray tails

$$\mathcal{R} := \{f^n(\gamma_z) : z \in S_I \text{ and } 0 \leq n \leq N(z)\},$$

which has finitely many elements as by (PF), $\#S_I$ is finite. Note that if γ is a ray tail, then by definition $\lim_{t\to\infty} f(\gamma(t)) = \infty$, and in particular, there exists a constant $R(\gamma) > 0$ such that $\gamma \setminus \mathbb{D}_{R(\gamma)} \subset \mathcal{T}_{\mathbb{D}_{R_0}}$. Since $\#\mathcal{R} < \infty$, there exists a finite constant

$$R_1 := \max_{\gamma \subset \mathcal{R}} R(\gamma).$$

Thus, for all $\gamma \in \mathcal{R}$, $\gamma \setminus \mathbb{D}_{R_1} \subset \mathcal{T}_{\mathbb{D}_{R_0}}$. Let us define tracts for f with respect to $\mathbb{D}_{R_1}$, that is, $\mathcal{T}_{\mathbb{D}_{R_1}} := f^{-1}(\mathbb{C} \setminus \mathbb{D}_{R_1})$. We can assume without loss of generality that $\mathbb{D}_{R_0} \subset \mathbb{D}_{R_1}$, since otherwise we can replace R_1 by a bigger constant. Thus, it holds that $\mathcal{T}_{\mathbb{D}_{R_1}} \subset \mathcal{T}_{\mathbb{D}_{R_0}}$, and by construction, for all $n \geq 0$ and $z \in S_I$, $f^n(\gamma_z) \subset (\mathbb{D}_{R_1} \cup \mathcal{T}_{\mathbb{D}_{R_0}})$. Consequently, by Proposition 2.19(3), we can choose a curve $\tilde{\delta} \subset \mathbb{C} \setminus (\overline{\mathcal{T}}_{\mathbb{D}_{R_0}} \cup \mathbb{D}_{R_1})$ connecting a point in $\partial \mathbb{D}_{R_1}$ to infinity. In particular,

$$f^n(\gamma_z) \cap \tilde{\delta} = \emptyset \quad \text{for all} \quad z \in S_I \quad \text{and} \quad n \geq 0. \tag{4.2.2}$$

Note that (4.2.2) is equivalent to

$$\bigcup_{z \in S_I} \gamma_z \ \cap \ \bigcup_{n \geq 0} f^{-n}(\tilde{\delta}) = \emptyset.$$

However, if we defined a set W_{-1} as $\mathbb{C} \setminus (\overline{\mathbb{D}}_{R_1} \cup \tilde{\delta})$, then W_{-1} would not satisfy the property on connected components of its preimages specified in the statement, since the curves $\{\gamma_z\}_{z \in S_I}$ could a priori intersect some preimage of $\partial \mathbb{D}_{R_1}$. Note that such property is equivalent to saying that, for some appropriate $\overline{D}$ and δ, if $f^n(z) \in \mathbb{C} \setminus (\overline{D} \cup \delta)$ for some $z \in S_I$ and $n \geq 1$, then $f^n(\gamma_z) \subset \mathbb{C} \setminus (\overline{D} \cup \delta)$. Hence, in order to define W_{-1} satisfying the property we are looking for, by (4.2.2), it suffices to find a domain $D \supset \mathbb{D}_{R_1}$ and a curve $\delta \subset \tilde{\delta}$ such that $\overline{D} \cap \delta$ is a single point and so that

$$\text{if} \quad z \in S_I \text{ and } f^n(z) \in (\mathbb{C} \setminus \overline{D}) \text{ for some } n \in \mathbb{N}, \quad \text{then} \quad f^n(\gamma_z) \subset (\mathbb{C} \setminus \overline{D}). \tag{4.2.3}$$

Thus, our next aim is to find a domain D for which (4.2.3) holds.

Arguing as before, by definition of ray tail, for each $z \in S_I$ there exists a constant $M(z) \in \mathbb{N}$ such that $f^m(\gamma_z) \subset (\mathbb{C} \setminus \overline{\mathbb{D}}_{R_1})$ for all $m \geq M(z)$. Hence, there exists a constant $Q \geq R_1$ such that

$$\mathbb{D}_Q \supset \{f^n(z) : \ z \in S_I \ \text{ and } \ 0 \leq n \leq M(z)\} =: \mathcal{P},$$

where the set $\mathcal{P}$ has only finitely many elements. As before, by definition of ray tail, only finitely many rays that are of the form $f^m(\gamma_z)$ for some $m \geq M(z)$ and $z \in S_I$ intersect $\mathbb{D}_Q$. Hence, we can find a domain D such that $\mathbb{D}_{R_1} \cup \mathcal{P} \subset D \subset \mathbb{D}_Q$ and so that $D \cap f^m(\gamma_z) = \emptyset$ for all $m \geq M(z)$ and $z \in S_I$. This means that since $\mathcal{P} \subset D$, the hypothesis in (4.2.3) can only hold for $f^m(z)$ with $m \geq M(z)$, but by construction and since $\mathcal{T}_D \subset T_{\mathbb{D}_{R_1}}$, the thesis in (4.2.3) always holds for these cases. Thus, (4.2.3) is always true for our choice of D. Defining δ

as the unbounded connected component of $\tilde{\delta} \setminus D$, the proof of the first part of the statement is concluded. The fact that for any connected component τ of $f^{-n}(W_{-1})$ it holds that $(P(f) \setminus I(f)) \cap \tau = \emptyset$, is a consequence of $(P(f) \setminus I(f)) \subset \mathbb{D}_{R_1} \subset D$ and $(P(f) \setminus I(f))$ being forward-invariant. $\square$

We are now ready to define the basic objects of this section.

Definition 4.17 (Fundamental hands)**.** Under the assumptions of Proposition 4.16, let W_{-1} be the domain from (4.2.1). Then, for each $n \geq 0$ define inductively

$$X_n := \bigcup_{z \in W_{n-1} \cap S_I} \gamma_z \qquad \text{and} \qquad W_n := f^{-1}\left(W_{n-1} \setminus X_n\right).$$

For every $n \geq 0$, each connected component of W_n is called a *fundamental hand* of level n.

That is, in a rough sense, we take successive preimages of W_{-1}, removing at each step some curves in $\{\gamma_z\}_{z \in S_I}$ whenever a point in S_I belongs to some component of a preimage. In particular, $X_0 = \emptyset$ and fundamental hands of level 0 are fundamental domains for f. The choice of W_{-1} in Proposition 4.16 has been made so that the following basic properties of fundamental hands hold:

Proposition 4.18 (Facts about fundamental hands)**.** *Fundamental hands are unbounded, simply connected, and any two of the same level are pairwise disjoint. Moreover, each fundamental hand of level $n > 1$ is mapped univalently under f to a fundamental hand of level $n - 1$.*

Proof. We prove all facts simultaneously using induction on n. For $n = 0$, fundamental hands are fundamental domains, and so the statement follows by Definition 2.18 and Proposition 2.19. Let us assume that the statement holds for some $n - 1 \in \mathbb{N}$, and we shall see that it holds for n. Let τ be a fundamental hand of level n. Then, by definition, its image $f(\tau)$ is contained in a fundamental hand $\tilde{\tau}$ of level $n - 1$, and

$$\partial\tilde{\tau} \subset \Big(f^{-n}(\partial W_{-1}) \cup \bigcup_{\substack{0<i<n \\ z \in S_I}} f^{-i}(\gamma_z)\Big). \tag{4.2.4}$$

By Proposition 4.16, X_n does not intersect $f^{-n}(\partial W_{-1})$. Since by (4.2.4) all other connected components that might form $\partial\tilde{\tau}$ are preimages of ray tails, and thus in $I(f)$, $X_n \cap (\partial\tilde{\tau} \setminus f^{-n}(\partial W_{-1}))$ must be simply connected, since otherwise there would be a domain enclosed by pieces of ray tails that escapes uniformly to infinity, contradicting that $I(f)$ has empty interior for functions in class $\mathcal{B}$, see

Theorem 2.17. Thus, by this and using the inductive hypothesis, $\tilde{\tau} \setminus X_n$ is an unbounded, simply connected domain. Since f is an open map, the same holds for τ.

In order to see that $f|_\tau$ is injective, note that all singular values contained in W_{n-1} also belong to X_n, and hence $\tau \cap \operatorname{Crit}(f) = \emptyset$. This implies that the restriction $f|_\tau$ is a covering map, and all inverse branches of f in the domain $f(\tau)$ can be continued. Moreover, as we have seen in the previous paragraph, $f(\tau) = \tilde{\tau} \setminus X_n$ is simply connected. This implies that all arcs in $f(\tau)$ with fixed endpoints are homotopic, and hence, by the Monodromy Theorem (see [Ahl78, Theorem 2, p.295]), given any two homotopic curves in $f(\tau)$, for an inverse branch of f defined in a neighbourhood of their starting endpoint, all its analytic continuations along the curves lead to the same values at the terminal endpoint, and so $f|_\tau$ is injective. By the inductive hypothesis, fundamental hands of level $n-1$ are pairwise disjoint, and since fundamental hands of level n are the connected components of the preimages of subsets of those hands, they are also pairwise disjoint. □

4.19 (Fixing external addresses for f)**.** Let $f \in \mathcal{B}$ be a criniferous function such that $J(f) \cap \operatorname{AV}(f) = \emptyset$ and satisfying (PF), and suppose that fundamental hands have been defined for f. Then, we define external addresses for f as sequences of fundamental hands of level 0, since in particular these are fundamental domains. As usual, we denote by $\operatorname{Addr}(f)$ the set of admissible external addresses for f, see Definition 2.21.

In the next proposition, that will serve us as an auxiliary result to prove Proposition 4.22, we show that fundamental hands of any level always intersect at least one fundamental domain and that this intersection is unbounded. As a consequence, we obtain that fundamental hands contain Julia constituents.

Proposition 4.20 (Fundamental hands contain Julia constituents)**.** *Under the assumptions of 4.19, let τ be a fundamental hand of level n for some $n \geq 0$. Then, there exists at least one fundamental domain F_1 such that $\tau \cap F_1$ is unbounded. Moreover, if $J_{\underline{\omega}} \subset \tau \setminus X_n$ for some $\underline{\omega} \in \operatorname{Addr}(f)$, then for each fundamental domain F_0, there exists a unique fundamental hand $\tilde{\tau}$ of level $n+1$ such that $f(\tilde{\tau}) \subset \tau$ and $J_{F_0\underline{\omega}} \subset F_0 \cap \tilde{\tau}$. In particular, there is $\underline{s} \in \operatorname{Addr}(f)$ with $J_{\underline{s}} \subset \tau$.*

Proof. In order to prove the first claim, we proceed by induction on the level n of the hand τ. For $n = 0$, τ is a fundamental domain and so the first claim is trivial. Suppose that it is true for some $n-1 \in \mathbb{N}$. In order to see that it also holds for n, note that by definition of fundamental hands, $f(\tau) \subset \tau_2$, where τ_2 is a fundamental hand of level $n-1$. Then, by the inductive hypothesis, there exists at least one fundamental domain F_1 such that $F_1 \cap \tau_2$ is unbounded. Let D be the domain containing $S(f)$ in the definition of fundamental domains, i.e., provided by Proposition 4.16, and let W_{-1} be the set from (4.2.1). Moreover, let $\overset{\infty}{F_1}$ be the unbounded connected component of $F_1 \setminus \overline{D}$, and let U be the unbounded connected component of $\overset{\infty}{F_1} \cap (f(\tau) \setminus \overline{D})$. Then, since by Proposition 4.18 $f|_\tau$ maps to $f(\tau)$ bijectively and since $U \subset W_{-1}$, there exists a unique fundamental domain F_0 containing the unbounded set $f^{-1}(U) \cap \tau$, and so we have proved the first claim.

For second claim, let $\underline{\omega} = F_1 F_2 \ldots \in \operatorname{Addr}(f)$ be as in the statement. In particular, $J_{\underline{\omega}} \subset \overset{\infty}{F_1}$. Then, for any other fundamental domain F_0, by the same argument as before, $f^{-1}(\overset{\infty}{F_1}) \cap F_0$ is an unbounded set that by definition contains $J_{F_0\underline{\omega}}$. In particular, by definition of fundamental hands and since they are pairwise disjoint, there is a unique fundamental hand $\tilde{\tau} \subset f^{-1}(\tau \setminus X_n)$ of level $n+1$ that contains $J_{F_0\underline{\omega}}$, as we wanted to show.

Finally, we construct $\underline{s} \in \operatorname{Addr}(f)$ such that $J_{\underline{s}} \subset \tau =: \tau_0$. Note that for each $0 \leq j \leq n$, $f^j(\tau) \subset \tau_j$ for some fundamental hand τ_j of level $n-j$. In particular, by the first part of the proposition, for each fundamental hand τ_j there exists a fundamental domain F_j such that $\tau_j \cap F_j$ is unbounded. Recall that the curves $\{\gamma_z\}_{z \in S_I}$ that form the sets X_j are canonical tails. Then, by Propositions 4.8 and 4.10 together with Observation 4.13, each γ_z contains $J^\infty_{\underline{\alpha}}$ for exactly one $\underline{\alpha} \in \operatorname{Addr}(f)$, and consequently, by (2.4.3), the same holds for each of the canonical tails in

$$\mathcal{R} := \{f^j(\gamma_z) : z \in S_I \text{ and } 0 \leq j \leq n\}.$$

The set $\mathcal{R}$ has finite cardinality, and thus, so does the set

$$\operatorname{Addr}(\mathcal{R}) := \{\underline{s} \in \operatorname{Addr}(f) : J^\infty_{\underline{s}} \subset \gamma \text{ for some } \gamma \in \mathcal{R}\}.$$

Since $\operatorname{Addr}(f)$ is an uncountable set, we can choose any bounded $\underline{\alpha} \in \operatorname{Addr}(f)$ such that $\underline{s} := F_0 F_1 \ldots F_n \underline{\alpha} \notin \operatorname{Addr}(\mathcal{R})$. Then, since $\underline{s}$ is also a bounded address, by Theorem 2.23 $J_{\underline{s}} \neq \emptyset$ and so $\underline{s} \in \operatorname{Addr}(f)$. By construction, $J^\infty_{\underline{s}} \cap X_j = \emptyset$ for all $0 \leq j \leq n$, and in particular, $J_{F_n\underline{\alpha}} \subset \tau_n \setminus X_n$. Then, by the second part of the proposition, $J_{F_{n-1}F_n\underline{\alpha}} \subset \tau_{n-1} \cap F_{n-1}$. Iterating this argument $n-2$ further times, we see that $J_{\underline{s}} \subset \tau$. □

As discussed at the beginning of the section, we are interested in finding neighbourhoods of canonical tails on which inverse branches are well-defined. These neighbourhoods will be provided by images of closures of fundamental hands. We first fix a choice of signed addresses and canonical tails:

4.21 (Fixing indexed canonical tails for f)**.** Let $f \in \mathcal{B}$ be a criniferous function such that $J(f) \cap \mathrm{AV}(f) = \emptyset$ and satisfying (PF), and suppose that fundamental hands and $\mathrm{Addr}(f)$ have been defined for f according to 4.19. Then, we define the set of signed addresses $\mathrm{Addr}(f)_\pm$ following Definition 4.3. Moreover, let

$$\mathcal{C} := \left\{ \gamma^n_{(\underline{s},*)} : n \geq 0 \text{ and } (\underline{s},*) \in \mathrm{Addr}(f)_\pm \right\}$$

be a set of canonical tails provided by Proposition 4.8 for any valid initial configuration. In particular, one can always use the configuration $\{J^\infty_{\underline{s}}\}_{\underline{s}\in\mathrm{Addr}(f)}$, see Observation 4.7.

Under the conditions in 4.21, in the next proposition we show that each canonical tail $\gamma^n_{(\underline{s},*)} \in \mathcal{C}$ belongs to the closure of at least one and at most two fundamental hands of level n. In addition, we assign to each canonical tail a fundamental hand that will allow us to define the desired inverse branches in the following Lemma 4.23. Recall that we have shown in Proposition 4.20 that given any fundamental hand, we can find a Julia constituent contained in it.

Proposition 4.22 (Fundamental hands for canonical tails)**.** *Let $f \in \mathcal{B}$ be a criniferous function such that $J(f) \cap AV(f) = \emptyset$ and satisfying (PF), and let $\mathcal{C}$ be the set in 4.21. For each $\gamma^n_{(\underline{s},*)} \in \mathcal{C}$, exactly one of the following holds:*

(A) There exists a unique fundamental hand τ of level n such that $\gamma^n_{(\underline{s},*)} \subset \overline{\tau}$. We denote

$$\tau_n(\underline{s},*) := \tau \cup \gamma^n_{(\underline{s},*)}.$$

(B) The curve $\gamma^n_{(\underline{s},*)}$ belongs to the boundary of exactly two fundamental hands τ and $\tilde{\tau}$ of level n. Let $\underline{\upsilon}, \underline{\omega} \in \mathrm{Addr}(f)$ such that $J^\infty_{\underline{\upsilon}} \subset \tau$ and $J^\infty_{\underline{\omega}} \subset \tilde{\tau}$. Then we denote

$$\tau_n(\underline{s},*) := \begin{cases} \tau \cup \gamma^n_{(\underline{s},*)} & \text{if } [\underline{\upsilon},\underline{s},\underline{\omega}]_\ell \text{ and } * = - \quad \text{or} \quad [\underline{\omega},\underline{s},\underline{\upsilon}]_\ell \text{ and } * = + \\ \tilde{\tau} \cup \gamma^n_{(\underline{s},*)} & \text{otherwise,} \end{cases} \tag{4.2.5}$$

where "$[\cdot]_\ell$" is the cyclic order in addresses defined in (2.4.8).

Remark. The definition of $\tau_n(\underline{s},*)$ in case (B) is independent of the choice of addresses $\underline{\upsilon}, \underline{\omega} \in \mathrm{Addr}(f)$ such that $J^\infty_{\underline{\upsilon}} \subset \tau$ and $J^\infty_{\underline{\omega}} \subset \tilde{\tau}$. To see this, note that by definition of fundamental hands, the connected component T of $\partial\tau$ that contains $J^\infty_{\underline{s}}$ separates the plane, and in particular τ and $\tilde{\tau}$ lie in different components of $\mathbb{C} \setminus T$. Moreover, T must contain a (preimage of a) critical point, and so

$J^\infty_{\underline{\alpha}} \subset T$ for some other $\underline{\alpha} \in \operatorname{Addr}(f)$. Then, we can define a linear order "<" in $\operatorname{Addr}(f)$ by cutting $\underline{\alpha}$. In particular, either $\underline{\upsilon} < \underline{s} < \underline{\omega}$ or $\underline{\upsilon} > \underline{s} > \underline{\omega}$ for all pairs of addresses $\underline{\upsilon}$ and $\underline{\omega}$ so that $J^\infty_{\underline{\upsilon}} \subset \tau$ and $J^\infty_{\underline{\omega}} \subset \tilde{\tau}$, and using the equivalence (2.4.9), the claim follows.

Proof of Proposition 4.22. Firstly, recall that by Observation 4.14, each canonical tail belongs to a curve $\gamma^m_{(\underline{\alpha},\star)}$ for some $m \geq 0$ and $(\underline{\alpha},\star) \in \operatorname{Addr}(f)_\pm$. In particular, this is the case for the canonical tails $\{\gamma_z\}_{z\in S_I}$ we chose in the definition of fundamental hands. We remark that the curves in $\{\gamma_z\}_{z\in S_I}$ might not be pairwise disjoint, but since $f \in \mathcal{B}$, $I(f)$ has empty interior (Theorem 2.17), and thus each connected component of $\mathbb{C} \setminus \bigcup_{z\in S_I} \gamma_z$ is simply connected. Then, the boundaries of fundamental hands are either connected components of preimages of curves in $\{\gamma_z\}_{z\in S_I}$, or preimages of the boundary of the set W_{-1} from Proposition 4.16. In particular, by Propositions 4.10 and 4.16, each curve $\gamma^n_{(\underline{s},*)}$ might only intersect boundaries of fundamental hands at (preimages of) critical points or might share with them a segment between any two of those preimages.

Let us fix a curve $\gamma^n_{(\underline{s},*)} \in \mathcal{C}$. Since by Propositions 4.8 and 4.10 together with Observation 4.13, each curve in $\{\gamma_z\}_{z\in S_I}$ contains a curve $\gamma^0_{\underline{\alpha}}$ for exactly one address $\underline{\alpha} \in \operatorname{Addr}(f)$, and the same holds for the unbounded components of $f^{-n}(\gamma_z)\setminus\operatorname{Crit}(f)$ for all $n \in \mathbb{N}$, the curve $\gamma^0_{(\underline{s},*)}$ is either totally contained in a hand of level n, or belongs to the boundary of two such hands whenever $f^i(\gamma^0_{(\underline{s},*)}) \subset X_i$ for some $1 \leq i \leq n$. We subdivide the proof into these two cases:

- Suppose that $\gamma^0_{(\underline{s},*)} \subset \tau$, where τ is a hand of level n. If $\gamma^n_{(\underline{s},*)} \subset \tau$, then case (A) holds. Otherwise, let $x \in \gamma^n_{(\underline{s},*)}$ be the point of greatest potential that also belongs to $\partial\tau$. By Propositions 4.16 and 4.10, x must be a (preimage of a) critical point, and there must be at least two canonical tails in $\partial\tau$ that also contain x. Recall that bristles are defined for canonical tails as either the successor or predecessor (bounded) segment in the circular order of segments around x. See Figure 8. Hence, the bristles of the unbounded component of $\gamma^n_{(\underline{s},*)} \setminus \{x\}$ must lie between this curve and two unbounded components of $\partial\tau\setminus\{x\}$ that contain x as an endpoint. Thus, these bristles belong to either $\partial\tau$ or τ. If the bounded component of $\gamma^n_{(\underline{s},*)} \setminus \{x\}$ does not contain any other (preimage of a) critical point, then we are done. Otherwise, $\gamma^n_{(\underline{s},*)}$ might intersect another boundary component of $\partial\tau$ in a point y, and by the same argument, the bristles of the corresponding unbounded component of $\gamma^n_{(\underline{s},*)} \setminus \{y\}$ must again lie in $\overline{\tau}$. Thus, case (A) holds for $\gamma^n_{(\underline{s},*)}$.

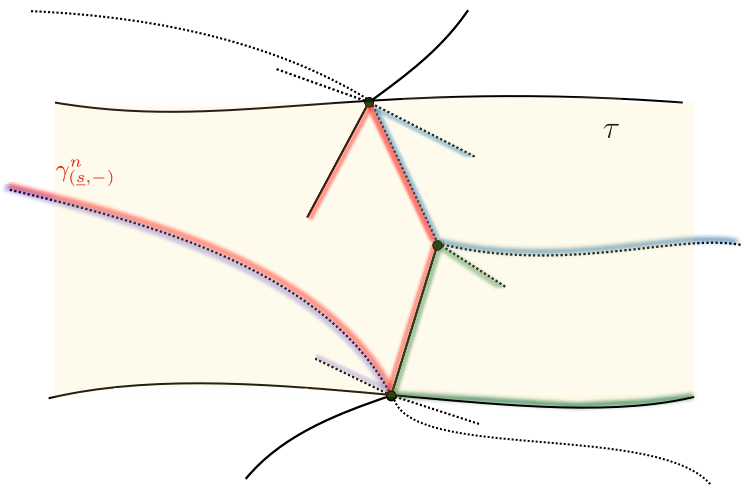

Figure 8: Example of a curve $\gamma^n_{(\underline{s},-)}$ contained in a fundamental hand τ of level n. The boundary of τ is represented with black continuous lines and is formed by ray tails. Dotted lines show other pieces of ray tails. Canonical tails that overlap with $\gamma^n_{(\underline{s},-)}$ are displayed in different colours.

- Suppose that $\gamma^0_{(\underline{s},*)} \subset \partial\tau \cap \partial\tilde{\tau}$, where τ and $\tilde{\tau}$ are two hands of level n. Let $x \in \gamma^n_{(\underline{s},*)}$ be the point of smallest potential that also belongs to $\partial\tau \cap \partial\tilde{\tau}$. If x is the endpoint of $\gamma^n_{(\underline{s},*)}$, case (B) holds and we are done. Otherwise, by the same argument as before, x must be a (preimage of a) critical point, and so, the continuation of $\gamma^n_{(\underline{s},*)}$ towards points of smaller potential than the one of x, is a nested sequence of left or right bristles. By minimality of x, the bristle that contains x can no longer be in the boundary of both τ and $\tilde{\tau}$, so it is either in the interior or in the boundary of only one of them. Then, from there we can argue as in the previous case and we see that case (A) holds. □

Finally, we use the sets $\tau_n(\underline{s},*)$ from the previous proposition to define, for any given signed address $(\underline{s},*)$ and $n > 0$, an inverse branch $\tilde{f}$ of f in a (not necessarily open) neighbourhood U of $\gamma^{n-1}_{(\sigma(\underline{s}),*)}$ such that $\tilde{f}(U) \supset \gamma^n_{(\underline{s},*)}$. Moreover, the definition and study of the concepts introduced in this section is justified in the following property, that will follow from the next lemma: *the neighbourhood U above is defined so that there exists an interval of addresses $\mathcal{I} \ni (\underline{s},*)$ in the cyclic order topology of* $\mathrm{Addr}(f)_\pm$*, see* (4.1.3)*, so that for all $(\underline{\alpha},\star) \in \mathcal{I}$, $\gamma^{n-1}_{(\sigma(\underline{\alpha}),\star)} \subset U$ and $\gamma^n_{(\underline{\alpha},\star)} \subset \tilde{f}(U)$.* In other words, we provide an explicit statement of the idea of convergence of canonical tails in terms of their addresses noted in Observation 4.15.

Lemma 4.23 (Inverse branches for canonical tails)**.** *Under the conditions of Proposition 4.22, for each $n \geq 0$ and $(\underline{s},*) \in \mathrm{Addr}(f)_\pm$:*

(1) There exists an open interval of signed addresses $\mathcal{I}^n(\underline{s},*) \ni (\underline{s},*)$ such that

$$\tau_n(\underline{\alpha},\star) \subseteq \tau_n(\underline{s},*) \quad \textit{for all} \quad (\underline{\alpha},\star) \in \mathcal{I}^n(\underline{s},*).$$

(2) If $n \geq 1$, the restriction $f|_{\tau_n(\underline{s},*)}$ is injective and maps to $\tau_{n-1}(\sigma(\underline{s}),*)$.

Hence, for all $n \geq 1$, we can define the inverse branch

$$f^{-1,[n]}_{(\underline{s},*)} := \left(f|_{\tau_n(\underline{s},*)}\right)^{-1} : f(\tau_n(\underline{s},*)) \longrightarrow \tau_n(\underline{s},*). \tag{4.2.6}$$

Proof. We start by showing (2). Recall that by Proposition 4.8, $f|_{\gamma^n_{(\underline{s},*)}}$ maps injectively to $\gamma^{n-1}_{(\sigma(\underline{s}),*)}$, and by Proposition 4.18, $f|_{\mathrm{int}(\tau_n(\underline{s},*))}$ maps injectively into a fundamental hand of level $n-1$. Suppose for the sake of contradiction that there exist $x \in \mathrm{int}(\tau_n(\underline{s},*)) \setminus \gamma^n_{(\underline{s},*)}$ and $y \in \gamma^n_{(\underline{s},*)} \setminus \mathrm{int}(\tau_n(\underline{s},*))$ so that $f(x) = f(y)$. In particular, $f(y) \in (X_n \cup \partial f(\tau_n(\underline{s},*)))$, but this would contradict $x \in \mathrm{int}(\tau_n(\underline{s},*))$. Thus, $f|_{\tau_n(\underline{s},*)}$ is injective.

If $\gamma^0_{(\underline{s},*)} \subset \mathrm{int}(\tau_n(\underline{s},*))$, then case (A) in Proposition 4.22 must occur for both $\gamma^n_{(\underline{s},*)}$ and $\gamma^{n-1}_{(\sigma(\underline{s}),*)}$. Then, since f is continuous, $f(\tau_n(\underline{s},*)) \subset \tau_{n-1}(\sigma(\underline{s}),*)$ and (2) follows. Otherwise, $\gamma^0_{(\underline{s},*)} \subset \partial\tau_n(\underline{s},*) \cap \partial\tilde{\tau}$ for $\tilde{\tau}$ another hand of level n. If $\gamma^0_{(\underline{s},*)} \subset f^{-1}(X_n)$, then $f(\gamma^0_{(\underline{s},*)})$ is totally contained in a fundamental hand of level $n-1$, namely $\tau_{n-1}(\sigma(\underline{s}),*)$. Then, by continuity of f, both $\tau_n(\underline{s},*)$ and $\tilde{\tau}$ are mapped under f to $\tau_{n-1}(\sigma(\underline{s}),*)$, and (2) follows. Hence, we are left to study the case when $\gamma^0_{(\underline{s},*)} \not\subseteq f^{-1}(X_n)$, which implies that $\gamma^0_{(\sigma(\underline{s}),*)}$ must also belong to $\partial\tau_{n-1}(\sigma(\underline{s}),*)$. By Proposition 4.20, we can choose a pair of addresses $\underline{a},\underline{b} \in \mathrm{Addr}(f)$ such that $\gamma^0_{\underline{a}} \subset \tau_n(\underline{s},*)$ and $\gamma^0_{\underline{b}} \subset \tilde{\tau}$. In particular, by Proposition 4.22 it must occur that $\tau_n(\underline{a},*) = \tau_n(\underline{s},*)$ and $\tau_n(\underline{b},*) = \tilde{\tau}$. We may assume without loss of generality that $[\underline{a},\underline{s},\underline{b}]_\ell$ holds, since the case when $[\underline{b},\underline{s},\underline{a}]_\ell$ does is analogous. Note that by (2.4.9), this is equivalent to $[\gamma^0_{\underline{a}},\gamma^0_{\underline{s}},\gamma^0_{\underline{b}}]_\infty$, and since by continuity of f the circular order at infinity of these curves is preserved under iteration of f, by (2.4.3), it holds that $[f(\gamma^0_{\underline{a}}), f(\gamma^0_{\underline{s}}), f(\gamma^0_{\underline{b}})]_\infty$. Moreover, by definition of valid initial configurations, $f(\gamma^0_{\underline{\alpha}}) \subset \gamma^0_{\sigma(\underline{\alpha})}$ for all $\underline{\alpha} \in \mathrm{Addr}(f)$, and hence it holds $[\gamma^0_{\sigma(\underline{a})},\gamma^0_{\sigma(\underline{s})},\gamma^0_{\sigma(\underline{b})}]_\infty$, and thus

$$[\sigma(\underline{a}),\sigma(\underline{s}),\sigma(\underline{b})]_\ell. \tag{4.2.7}$$

Recall that the sign $* \in \{-,+\}$ is preserved in the curve $\gamma^n_{(\underline{s},*)}$ under the action of f, as $f(\gamma^n_{(\underline{s},*)}) = \gamma^{n-1}_{(\sigma(\underline{s}),*)}$. Then, (4.2.7) together with continuity of f implies that if case (B) in Proposition 4.22 holds for either $\gamma^n_{(\underline{s},*)}$, $\gamma^{n-1}_{(\sigma(\underline{s}),*)}$, or both curves, then $\tau_n(\underline{s},*)$ and $\tau_{n-1}(\sigma(\underline{s}),*)$ are chosen so that (2) holds.

We prove (1) by induction on n. If $n = 0$, then for any $(\underline{s}, *) \in \mathrm{Addr}(f)_\pm$, $\tau_0(\underline{s}, *) = F_0$ for some fundamental domain F_0. By Theorem 2.23, we can choose a pair of bounded addresses $\underline{a}, \underline{b}$ whose first entry is $\overset{\infty}{F_0}$ and so that $[\underline{a}, \underline{s}, \underline{b}]_\ell$, and define $\mathcal{I}^0(\underline{s}, *) := ((\underline{a}, -), (\underline{b}, +))$. Then, $\gamma^0_{(\underline{\alpha}, \star)} \subset F_0$ for all addresses $(\underline{\alpha}, \star) \in \mathcal{I}^0(\underline{s}, *)$, and so $\tau_0(\underline{\alpha}, \star) = F_0$ and (1) follows.

Let us assume that the statement holds for all addresses in $\mathrm{Addr}(f)_\pm$ and for some $n - 1 \in \mathbb{N}$. Suppose that $\underline{s} = F_0 F_1 \ldots$ and note that by the inductive hypothesis, the interval $\mathcal{I}^{n-1}(\sigma(\underline{s}), *) \ni (\sigma(\underline{s}), *)$ is defined. If

$$\gamma^0_{(\sigma(\underline{s}), *)} \subset \mathrm{int}(\tau_{n-1}(\sigma(\underline{s}), *)) \setminus X_n, \tag{4.2.8}$$

we can choose $\mathcal{I}' \subseteq \mathcal{I}^{n-1}(\sigma(\underline{s}), *)$ such that $\gamma^0_{(\underline{\alpha}, \star)} \subset \mathrm{int}(\tau_{n-1}(\sigma(\underline{s}), *)) \setminus X_n$ for all $(\underline{\alpha}, \star) \in \mathcal{I}'$ and $(\sigma(\underline{s}), *) \in \mathcal{I}'$. Then, for each $(\underline{\alpha}, \star) \in \mathcal{I}'$, by Proposition 4.22, there exists a unique fundamental hand $\tilde{\tau}$ of level n such that $f(\tilde{\tau}) \subset \mathrm{int}(\tau_{n-1}(\sigma(\underline{s}), *))$ and $\gamma^0_{F_0\underline{\alpha}} \subset \tilde{\tau}$, and by continuity of f, the hand $\tilde{\tau}$ must equal $\tau_n(F_0\underline{\alpha}, \star) = \tau_n(\underline{s}, *)$. Hence, all the addresses in the set

$$S := \{(F_0\underline{\alpha}, \star) \in \mathrm{Addr}(f)_\pm : (\underline{\alpha}, \star) \in \mathcal{I}'\}$$

satisfy the property required. Moreover, $(\underline{s}, *) \in S$ and S is an interval of addresses, since by continuity, f preserves the order at infinity of extensions of level 0. More specifically, if $\mathcal{I}' = ((\underline{a}, *), (\underline{b}, \star))$ for some $\underline{a}, \underline{b} \in \mathrm{Addr}(f)$, then $S = ((F_0\underline{a}, *), (F_0\underline{b}, \star)) =: \mathcal{I}_n(\underline{s}, *)$.

Otherwise, if (4.2.8) does not hold for $\gamma^0_{(\sigma(\underline{s}), *)}$, then either $\gamma^0_{(\sigma(\underline{s}), *)} \subset X_n$, or $\gamma^0_{(\sigma(\underline{s}), *)} \subset \partial\tau_{n-1}(\sigma(\underline{s}), *)$. In the second case, the interval of addresses $\mathcal{I}^{n-1}(\sigma(\underline{s}), *)$ must be of the form

$$\mathcal{I}^{n-1}(\sigma(\underline{s}), *) = \begin{cases} ((\underline{s}, -), (\underline{a}, \star)) & \text{if } * = + \\ ((\underline{a}, \star), (\underline{s}, +)) & \text{if } * = - \end{cases}$$

for some $(\underline{a}, \star) \in \mathrm{Addr}(f)_\pm$. To see this, we might assume without loss of generality that $* = +$. Then, since $\gamma^0_{(\sigma(\underline{s}), -)} \in \partial\tau_{n-1}(\sigma(\underline{s}), *)$, any open interval of addresses containing $(\sigma(\underline{s}), -)$ would also have to contain signed addresses whose corresponding Julia constituents lie in another fundamental hand. Thus, $\mathcal{I}^{n-1}(\sigma(\underline{s}), *)$ must be an open interval containing $(\underline{s}, +)$ but not $(\underline{s}, -)$, and so must be of the form claimed[8]. Then, arguing as before, we can find a subinterval of addresses $\mathcal{I}' \subseteq \mathcal{I}^{n-1}(\sigma(\underline{s}), *)$ such that the curve $\gamma^0_{(\underline{\alpha},\star)} \subset \operatorname{int}(\tau_{n-1}(\sigma(\underline{s}), *)) \setminus X_n$ for all $(\underline{\alpha}, \star) \in \mathcal{I}' \setminus \{(\underline{s}, +)\}$. By the analysis of the previous case, if we let $\mathcal{I}' := ((\underline{s}, +), (\underline{b}, +))$, then for all addresses $(\underline{\alpha}, \star) \in ((F_0\underline{s}, +), (F_0\underline{b}, +))$, it holds that $\gamma^n_{(\underline{\alpha},*)} \subset \tau_n(\underline{\alpha}, \star) = \tau_n(\underline{s}, *)$. Then, for the statement to hold, we include the address $(\underline{s}, +)$ on the interval by defining $\mathcal{I}^n_{(\underline{s},*)} := ((\underline{s}, -), (F_0\underline{b}, +))$.

We are left to consider the case when $\gamma^0_{(\sigma(\underline{s}),*)} \subset X_n$. Note that by definition of fundamental hands, this implies that $\gamma^0_{(\underline{s},*)} \subset \partial\tau_n(\underline{s}, *)$. For the purposes of defining the desired interval, we can regard $\gamma^0_{(\sigma(\underline{s}),*)}$ as if it belonged to the boundary of its fundamental hand in order to apply the same reasoning as before. That is, we choose a subinterval $\mathcal{I}' \subset \mathcal{I}^{n-1}(\sigma(\underline{s}), *)$ of the form

$$\mathcal{I}' := \begin{cases} ((\underline{s}, -), (\underline{a}, \star)) & \text{if } * = + \\ ((\underline{a}, \star), (\underline{s}, +)) & \text{if } * = - \end{cases}$$

for some $\underline{a} \in \operatorname{Addr}(f)$ and proceed as in the previous case. and proceed as in the previous case. $\square$

We can now make explicit and justify the idea from the beginning of the section of finding for each $n \geq 0$ inverse branches of f^n defined on neighbourhoods of canonical tails satisfying certain properties:

Observation 4.24 (Chains of inverse branches)**.** Following Lemma 4.23, for each $n \geq 0$ and $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$, we denote

$$f^{-n}_{(\underline{s},*)} := \left(f^n|_{\tau_n(\underline{s},*)}\right)^{-1} : f^n(\tau_n(\underline{s}, *)) \to \tau_n(\underline{s}, *).$$

Then, by Lemma 4.23(2), the following chain of embeddings holds:

$$\tau_n(\underline{s}, *) \xhookrightarrow{f} \tau_{n-1}(\sigma(\underline{s}), *) \xhookrightarrow{f} \tau_{n-2}(\sigma(\underline{s}), *) \xhookrightarrow{f} \cdots \xhookrightarrow{f} \tau_0(\sigma^n(\underline{s}), *).$$

This means that we can express the action of $f^{-n}_{(\underline{s},*)}$ in $f^n(\tau_n(\underline{s}, *))$ as a composition of functions defined in (4.2.6). That is,

$$\tau_n(\underline{s}, *) \xleftarrow{f^{-1,[n]}_{(\underline{s},*)}} f(\tau_n(\underline{s}, *)) \xleftarrow{f^{-1,[n-1]}_{(\sigma(\underline{s}),*)}} f^2(\tau_n(\underline{s}, *)) \xleftarrow{f^{-1,[n-2]}_{(\sigma^2(\underline{s}),*)}} \cdots \xleftarrow{f^{-1,[1]}_{(\sigma^{n-1}(\underline{s}),*)}} f^n(\tau_n(\underline{s}, *)).$$

[8] See also Observation 4.53 for a discussion on open and closed intervals in the topology of $\operatorname{Addr}(f)_\pm$.

More precisely,

$$f^{-n}_{(\underline{s},*)} \equiv \left(f^{-1,[n]}_{(\underline{s},*)} \circ f^{-1,[n-1]}_{(\sigma(\underline{s}),*)} \circ \cdots \circ f^{-1,[1]}_{(\sigma^{n-1}(\underline{s}),*)} \right)\Big|_{f^n(\tau_n(\underline{s},*))}.$$

Moreover, combining this with Proposition 4.22 and Lemma 4.23, it follows that for all $(\underline{\alpha}, \star) \in \mathcal{I}^n(\underline{s}, *)$,

$$f^n(\tau_n(\underline{\alpha}, \star)) \subseteq f^n(\tau_n(\underline{s}, *)) \quad \text{ and } \quad f^{-n}_{(\underline{s},*)}\Big|_{f^n(\tau_n(\underline{\alpha},\star))} \equiv f^{-n}_{(\underline{\alpha},\star)}\Big|_{f^n(\tau_n(\underline{\alpha},\star))}.$$

4.3 Cantor bouquets and Julia sets

In this section, we provide a formal definition of the object called *Cantor bouquet*, which is homeomorphic to the Julia set of many transcendental entire functions, see Proposition 4.39 for examples. A common problem studied in the literature is that of trying to show that the Julia set of functions satisfying certain properties is a Cantor bouquet. Instead and conversely, in this section we assume for a function that its Julia set has this structure, and infer dynamical properties of the map. Our main goal is to prove Theorem 1.11, that is, to show that if $g \in \mathcal{B}$ is of disjoint type and its Julia set is a Cantor bouquet, then there is a way of "projecting" $J(g)$ in a continuous fashion to a subset of points whose orbit lies in a neighbourhood of infinity. More precisely, recall that for each $R > 0$ we denote

$$J_R(g) := \{z \in J(g) : |g^n(z)| \geq R \text{ for all } n \geq 1\}. \tag{4.3.1}$$

Then, we shall see that there exists a continuous map $\pi\colon J(g) \to J_R(g)$, defined in (4.3.3), such that for each $z \in J(g)$, both z and $\pi(z)$ lie in the same connected component of $J(g)$. We have been able to achieve this result thanks to the fact that, roughly speaking, the Cantor bouquet structure provides some control on "how much" the connected components of $J(g)$, which are curves, can "bend back and forth", see Proposition 4.26 and Figure 9. Compare to [RRRS11, Propositions 4.4 and 4.6], as a similar phenomenon occurs for connected components of $J(F)$ when F is a logarithmic transform satisfying a uniform head-start condition. In addition, Proposition 4.27 gathers together more general properties that all entire functions whose Julia set is a Cantor bouquet share.

Definition 4.25 (Straight brush, Cantor bouquet [BJR12, Definition 2.1])**.** A subset B of $[0, +\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is a *straight brush* if the following properties are satisfied:

- The set B is a closed subset of $\mathbb{R}^2$.

- For each $(x, y) \in B$, there is $t_y \geq 0$ so that $\{x : (x, y) \in B\} = [t_y, +\infty)$. The set $[t_y, +\infty) \times \{y\}$ is called the *hair* attached at y, and the point (t_y, y) is called its *endpoint*.

- The set $\{y : (x, y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$ there exist two sequences of hairs attached respectively at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n$, $\beta_n, \gamma_n \to y$ and $t_{\beta_n}, t_{\gamma_n} \to t_y$ as $n \to \infty$.

A *Cantor bouquet* is any subset of the plane that is ambiently homeomorphic[9] to a straight brush. A *hair* (resp. *endpoint*) of a Cantor bouquet is any preimage of a hair (resp. endpoint) of a straight brush under a corresponding ambient homeomorphism.

Remark. Any two straight brushes are ambiently homeomorphic, see [AO93], which in a broad sense means that the homeomorphism preserves the "vertical" order of the hairs in the brushes. In particular, the concepts of *hair* and *endpoint* of a Cantor bouquet are independent of the straight brush taken. Moreover, we note that even if we have referred informally to dynamic rays as hairs in the introduction, from then on, we have stopped using the word *hair*, instead reserving it for this context. In particular, hairs of Cantor bouquets are not necessarily dynamic rays. See Proposition 4.27.

A rather simple yet interesting property of Cantor bouquets, which will play a crucial role in our future arguments, is the following:

Proposition 4.26 (Jordan curves that hairs intersect at most once)**.** *Given a Cantor bouquet X, for each $R > 0$ there exists a bounded simply connected domain $S_R \Supset \mathbb{D}_R$ such that each hair of X intersects ∂S_R at most once.*

Proof. Let B be a straight brush and let $\psi : X \to B$ be the ambient homeomorphism in the definition of Cantor bouquet such that $\psi(X) = B$. Fix any $R \geq 0$. Then, since $\psi(\mathbb{D}_R)$ is a bounded set as $\psi(\overline{\mathbb{D}}_R)$ is the image of a compact set under a continuous function, $\psi(\mathbb{D}_R) \Subset \{(x, y) \in \mathbb{R}^2 : |x| < Q \text{ and } |y| < Q\} =: (-Q, Q)^2$ for some $Q \in \mathbb{Q}$. See Figure 9. By the choice of Q being a rational number, each hair of the brush B intersects the boundary of $(-Q, Q)^2$ in at most one point. Defining $S_R := \psi^{-1}((-Q, Q)^2)$, the proposition follows. $\square$

Whenever the Julia set of an entire function is a Cantor bouquet, we will informally refer to it as a *Julia Cantor bouquet*. We shall now see that without further assumptions, for an entire function f, having a Julia Cantor bouquet

[9] Two sets A and B in $\mathbb{R}^n$ are ambiently homeomorphic if there is a homeomorphism of $\mathbb{R}^n$ to itself that sends A onto B.

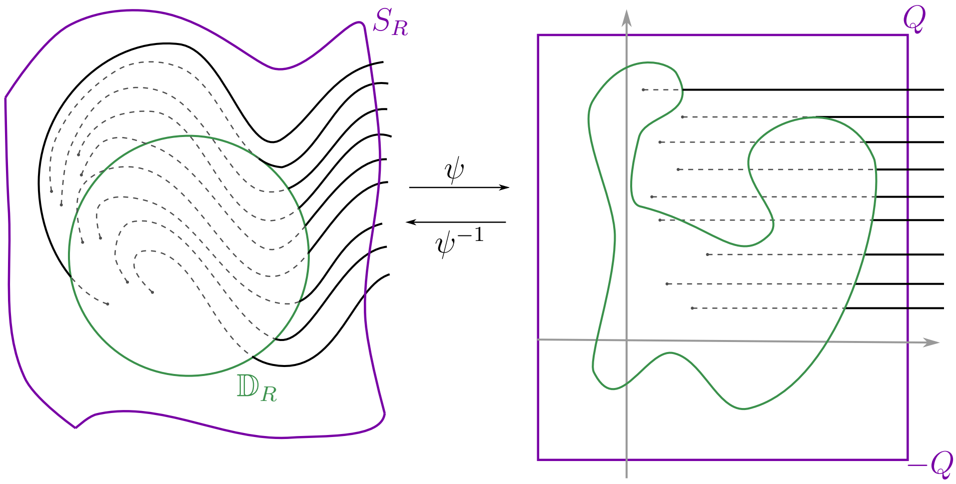

Figure 9: On the left, hairs of a Cantor bouquet intersecting a circle $\partial\mathbb{D}_R$, some of them multiple times. For each hair, dashes represent points with lower potential than that of the last point that intersects $\partial\mathbb{D}_R$. On the right, the image of the hairs to a straight brush under an ambient homeomorphism ψ. $[-Q, Q]^2$ is a square whose boundary the hairs intersect at most once, and $S_R := \varphi^{-1}((-Q, Q)^2)$.

already has interesting dynamical implications, in particular regarding the type of singular values that can occur in its Julia set. An asymptotic value a of f is *logarithmic* if there exists a neighbourhood U of a and a connected component V of $f^{-1}(U)$ such that $f\colon V \to U \setminus \{a\}$ is a universal covering map.

Proposition 4.27 (Properties of functions with Julia Cantor bouquets)**.** *Let g be an entire function and suppose that $J(g)$ is a Cantor bouquet. Then, the following hold:*

(A) $J(g)$ contains neither critical values nor logarithmic asymptotic values. In particular, if $S(g)$ is finite, then $S(g) \cap J(g) = \emptyset$.

(B) If η is a hair of $J(g)$, then $g(\eta)$ is a hair of $J(g)$.

(C) If $z \in J(g) \cap I(g)$ and γ is the (piece of) hair in $J(g)$ joining z to infinity, then $g^n|_\gamma \to \infty$ as $n \to \infty$ uniformly.

If in addition g is of disjoint type, then each hair of $J(g)$ is a dynamic ray together with its endpoint.

Proof. We start proving (A) by contradiction. Suppose that $J(g)$ contains a logarithmic asymptotic value z, which in particular is in a hair η. Then, by definition of logarithmic asymptotic value, $g^{-1}(\eta)$ contains a curve separating the plane, which contradicts the definition of $J(g)$ being a Cantor bouquet. Now suppose that there exists $z \in \mathrm{Crit}(g) \cap J(g)$ and let η be the hair that contains z. By continuity of g, $g(\eta)$ must be contained in a hair, say $\tilde{\eta}$. Then, since g acts as the map $w \mapsto w^{\deg(g,z)}$ locally in a neighbourhood of z, $J(g)$ being a Cantor bouquet, and hence a collection of hairs, has the following implications: it can only occur that $\deg(g,z) = 2$, z cannot be the endpoint of η and $g(z)$ must be the endpoint of $\tilde{\eta}$. For the same reason, if we denote by e and $\tilde{e}$ the respective endpoints of η and $\tilde{\eta}$, then $e \notin \operatorname{Crit}(g)$ and $g(e) = \tilde{e}$. We might assume without loss of generality that z is the point in $\operatorname{Crit}(g) \cap \eta$ with least potential, i.e., such that the restriction of η between e and z, that we denote by $\eta[e,z]$, does not contain any other critical point. We can make this assumption because the restriction of η between e and any critical point in η is compact, and thus, a critical point with minimal potential must exist. We have that $g(\eta[e,z]) \subset \tilde{\eta}$ and $g(e) = g(z) = \tilde{e}$. These two conditions can only be simultaneously fulfilled if $\eta[e,z]$ contains at least another critical point, which contradicts that by the minimality of z, all points between e and z are regular. Thus, we have shown that $J(g)$ cannot contain logarithmic asymptotic values nor critical values. Since for functions with finite singular set these are all types of points that $S(g)$ can contain, (A) is proved. In particular, we have shown in the proof of (A) that the endpoint of a hair must be mapped to the endpoint of the hair that contains its image under g, and so we have shown (B).

In order to prove (C), for each constant $R > 0$, let S_R be a bounded set provided by Proposition 4.26. Let $z \in J(g) \cap I(g)$ and let γ be the (piece of) hair joining z to infinity. Since $z \in I(g)$, there exists $N := N(R,z) \in \mathbb{N}$ such that

$$\bigcup_{n \geq N} g^n(z) \subset \mathbb{C} \setminus S_R. \tag{4.3.2}$$

This together with Proposition 4.26 implies that $g^n(\gamma) \subset \mathbb{C} \setminus S_R \subset \mathbb{C} \setminus \mathbb{D}_R$ for each $n \geq N$, since otherwise $g^n(\gamma)$ would intersect ∂S_R at least twice, contradicting Proposition 4.26. Hence, since this argument applies to all R, $g^n|_\gamma \to \infty$ uniformly as $n \to \infty$.

By definition, if $g \in \mathcal{B}$ is of disjoint type, then $P(g) \subset F(g)$. By this and since the image of each hair must lie in a hair, the restriction of g to any hair of $J(g)$ is injective. By [RG16, Theorem 5.8], each hair of $J(g)$ but at most its endpoint belongs to $I(g)$, and thus, by (C), each hair of $J(g)$ is a dynamic ray together with its endpoint. $\square$

As a consequence of the previous proposition, since the escaping set of any function in class $\mathcal{B}$ is contained in its Julia set (Theorem 2.17), we have the following corollary.

Corollary 4.28 (Julia Cantor bouquet implies criniferous)**.** *Any $f \in \mathcal{B}$ for which $J(f)$ is a Cantor bouquet is criniferous.*

Our next goal is to define for each disjoint type map g and each $R > 0$, a continuous function from $J(g)$ to $J_R(g)$ that acts like the identity map for all points in $J_R(g)$ of sufficiently large modulus. It follows from Proposition 4.27 that each hair of $J(g)$ contains an unbounded connected component in $J_R(g)$. Hence, one could think of defining the desired function as the identity in each of those unbounded components of hairs, and as the "projection" of the rest of points in a hair to the endpoint of the corresponding unbounded component. However, such function is not necessarily continuous: some hairs of $J(g)$ might be tangent to $\partial\mathbb{D}_R$, and hence the "projection" points of nearby hairs might be far. See Figure 9. We overcome this obstacle by considering for each hair the set of points whose orbit never meets a set S_R provided by Proposition 4.26. Then continuity is obtained, since tangencies between hairs and ∂S_R cannot occur. We formalize these ideas:

4.29 (Definition of the functions π_R)**.** Let $g \in \mathcal{B}$ be a disjoint type function for which $J(g)$ is a Cantor bouquet. In particular, note that any hyperbolic function for which $J(g)$ is a Cantor bouquet is of disjoint type: by the structure of $J(g)$ as a collection of pairwise disjoint curves, $\mathbb{C} \setminus J(g)$ is connected. We define a partial order relation "$\succeq$" in $J(g)$ by arranging the points on each hair according to their potentials. More specifically, if $J(g) = \psi^{-1}(B)$ for some straight brush B, for any pair of points z, w belonging to the same hair of $J(g)$, we say that

$$w \succeq z \iff \text{the real coordinate of } \psi(w) \text{ is greater than or equal to that of } \psi(z).$$

That is, $\psi(w)$ is in the same hair and "more to the right" than $\psi(z)$ in B. Or equivalently, if z, w are in the hair η and $z = \eta(t)$ and $w = \eta(t')$ for a parametrization of η, then $t' > t$. Note that the relation "$\succeq$" provides a total order when restricted to each hair of $J(g)$, and its definition is independent of the homeomorphism "ψ" chosen. As usual, $\prec, \preceq$ and $\succ$ denote the inverse and strict orders, i.e., $w \succeq z$ if and only if $z \preceq w$. The relation "$\succeq$" allows us to define the functions π_R: for each $R > 0$, let S_R be a domain given by Proposition 4.26 and let $J_R(g)$ be the set from (4.3.1). Then, we define the function $\pi_R \colon J(g) \to J_R(g)$ as

$$\pi_R(z) := \min\left\{ w \succeq z : \bigcup_{n \geq 0} g^n(w) \subset \mathbb{C} \setminus S_R \right\}. \tag{4.3.3}$$

That is, provided we show that π_R is well-defined (i.e., for each $z \in J(g)$ the set in the definition of $\pi_R(z)$ is non-empty and contains an absolute minimum), the function π_R acts as the identity for all those points whose forward orbit never meets S_R (and in particular belong to $J_R(g)$), while all other points are projected to the closest point "to their right" in their hair that is of the first type.

Observation 4.30 (Codomain of π_R)**.** For all $z \in J(g)$, $\pi_R(z) \in \mathbb{C} \setminus S_R \subset \mathbb{C} \setminus \mathbb{D}_R$, and so the codomain of π_R is indeed $J_R(g)$. Moreover, if $Q > R$ is a constant such that $S_R \Subset \mathbb{D}_Q$, then for any $w \in J_Q(g)$, $\pi_R(w) = w$, and hence $J_Q(g) \subset \pi_R(J(g))$.

In a rough sense, we shall see in the next theorem that for each $R > 0$, the function π_R from (4.3.3) is well-defined by showing that there exists a point on each hair of $J(g)$ to which all points "to its left" are projected, and in addition, when moving transversally to different hairs, their corresponding "projecting" points are "close enough" for the function π_R to be continuous. In particular, this theorem is a more precise version of Theorem 1.11 in Chapter 1.

Theorem 4.31 (Continuity of π_R)**.** *For each $R > 0$, let $\pi_R : J(g) \to J_R(g)$ be the function specified in 4.29. Then π_R is well-defined and continuous. In particular, for each hair η of $J(g)$, there is a point $z_\eta \in \eta$ such that $\pi_R(w) = z_\eta$ for all $w \prec z_\eta$, and $\pi_R(w) = w$ otherwise. Moreover, if g has a logarithmic transform $G : \mathcal{T} \to \mathbb{H}_{\log L}$ such that $\overline{\mathcal{T}} \subset \mathbb{H}_{\log L+8\pi}$ for some $L > 0$, then there exists a constant $M > R$ such that $\pi_R(z) \in \overline{A(M^{-1}|z|, M|z|)}$ for all $z \in J(g)$.*

Proof. Let us fix some $R > 0$. Seeking simplicity of notation, π_R will be denoted by π. For each $n \geq 0$, we define the functions $\pi_n : J(g) \to J_R(g)$ as

$$\pi_n(z) := \min \left\{ w \succeq z : \bigcup_{j=0}^{n} g^j(w) \subset \mathbb{C} \setminus S_R \right\}. \tag{4.3.4}$$

We are aiming to prove that the functions $\{\pi_n\}_{n\in\mathbb{N}}$ are well-defined, continuous and converge to the function π as n tends to infinity. Seeking clarity of exposition, we may assume without loss of generality that $J(g)$ is an embedding in $\mathbb{C}$ of a straight brush. Otherwise, we could prove continuity of the functions $\{\pi_n\}_{n\in\mathbb{N}}$ by showing that, after the usual identification of $\mathbb{C}$ and $\mathbb{R}^2$, the functions $\{(\psi^{-1} \circ \pi_n \circ \psi) \colon B \to B\}_{n\in\mathbb{N}}$ are continuous, where B is a straight brush and ψ is the corresponding ambient homeomorphism such that $\psi(B) = J(g)$. Thus, with this assumption made, the set S_R from Proposition 4.26 can be chosen to be of the form $S_R = (-Q, Q)^2$ for some $Q \in \mathbb{Q}$. Then, each hair of $J(g)$ is a horizontal half-line parallel to the real axis that by Proposition 4.26 intersects ∂S_R in at most one point belonging to the vertical line $\{z : \operatorname{Re} z = Q\}$.

For each $n \in \mathbb{N}$ and each hair η of $J(g)$, let

$$z_n := z_n(\eta) = \min\left\{z \in \eta : \bigcup_{j=0}^{n} g^j(z) \subset \mathbb{C} \setminus S_R\right\},$$

where the minimum value is taken with respect to the relation "$\succeq$" from 4.29. Note that each point z_n is indeed well-defined: by Proposition 4.27, for each $0 \leq j \leq n$, $g^j|_\eta$ is a bijection to a hair of $J(g)$, and by Proposition 4.26, $g^j(\eta) \cap \partial S_R$ consists of at most one point. Thus, since $\partial S_R \subset \mathbb{C} \setminus S_R$, z_n is either the endpoint of η, or z_n is the point in η with greatest potential such that $g^j(z_n) \in \partial S_R$ for some $j \leq n$. This allows us to conveniently express for each $n \in \mathbb{N}$ the action of the function π_n the following way:

$$\text{if } z \text{ is in the hair } \eta \subset J(g), \quad \text{then} \quad \pi_n(z) = \begin{cases} z_n(\eta) & \text{if } z \prec z_n(\eta), \\ z & \text{if } z \succeq z_n(\eta), \end{cases}$$

and thus, it follows that the functions π_n are well-defined. For each hair η, it is worth noting the following relation between the points in the set $\{z_n(\eta)\}_{n \geq 0}$:

Claim 1. For each hair η of $J(g)$ and all $n \geq 0$, $z_n(\eta) \preceq z_{n+1}(\eta)$.

Proof of claim 1. Since $g^n|_\eta$ maps bijectively to another hair, for any $z, w \in \eta$, $z \prec w$ if and only if $g^n(z) \prec g^n(w)$. In particular, using Proposition 4.26, if $g^{n+1}(z_n) \in \mathbb{C} \setminus S_R$ then $z_n = z_{n+1}$, and if on the contrary $g^{n+1}(z_n) \in S_R$, then $g^{n+1}(z_{n+1}) = g^{n+1}(\eta) \cap \partial S_R$, and thus $z_n \prec z_{n+1}$. △

In order to prove continuity of each function π_n, we will use that for "close enough" hairs, their corresponding "projection points z_n" are close. More precisely:

Claim 2. Let η be a hair of $J(g)$ for which $g^j(z_n(\eta)) \in \partial S_R$ for some $0 \leq j \leq n$. Then, there exists a constant $\alpha := \alpha_n(\eta) > 0$ such that if $\tilde{\eta}$ is another hair of $J(g)$ for which $B_\alpha(z_n(\eta)) \cap \tilde{\eta} \neq \emptyset$, then $z_n(\tilde{\eta}) \in B_\alpha(z_n(\eta))$.

Proof of claim 2. Seeking clarity of exposition, for the proof of this claim we use the notation $z_j := z_j(\eta)$ and $\tilde{z}_j := z_j(\tilde{\eta})$ for each $0 \leq j \leq n$. Let

$$0 \leq j_1 < j_2 < \ldots < j_k \leq n \tag{4.3.5}$$

be the sequence of iterates for which $g^{j_i}(z_n) \in \partial S_R$. In particular, by the assumption in the claim, the constant k in (4.3.5) is at least 1. It follows from Claim 1 that $z_j = z_n$ for all $j_1 \leq j \leq n$. Thus, our first goal is to find a neighbourhood A_1 of z_{j_1} such that if another hair $\tilde{\eta}$ intersects that neighbourhood, then its corresponding $\tilde{z}_{j_1}$ belongs to it. If $j_1 = 0$, since Q is a rational number and hairs of straight brushes are by definition at irrational heights, we can choose any $\epsilon_1 > 0$ such that $|Q - \mathrm{Im}(z_0)| < \epsilon_1$. If $\tilde{\eta}$ is a hair intersecting $B_{\epsilon_1}(z_0)$, since $\tilde{\eta}$ is a horizontal straight line, either $\tilde{z}_0$ is the single point in $\partial S_R \cap \tilde{\eta}$ whenever that intersection is non-empty, or $\tilde{z}_0$ is the endpoint of $\tilde{\eta}$. In any case $\tilde{z}_0 \in B_{\epsilon_1}(z_0)$, and so we define $A_1 := B_{\epsilon_1}(z_0)$.

If on the contrary $j_1 \neq 0$, since by definition $g^j(z_{j_1}) \in \mathbb{C} \setminus \overline{S}_R$ for all $j < j_1$, using that g is an open map, we can find $\delta > 0$ such that $g^j(B_\delta(z_{j_1})) \subset \mathbb{C} \setminus \overline{S}_R$ for all $0 \leq j < j_1$. For the same reason as before, we can choose some $\epsilon_1 > 0$ so that

$$B_{\epsilon_1}(g^{j_1}(z_{j_1})) \subset g^{j_1}(B_\delta(z_{j_1})) \quad \text{and} \quad |Q - \mathrm{Im}(g^{j_1}(z_{j_1}))| < \epsilon_1.$$

That is, we are choosing some ball centred at $g^{j_1}(z_{j_1})$ and contained in the j_1th-image of $B_\delta(z_{j_1})$ such that, by our assumption on S_R being the open square $(-Q, Q)^2$ and because $g^{j_1}(z_{j_1}) \in \partial S_R$, half of the ball lies in S_R. See Figure 10. Let A_1 be the connected component of the j_1-th preimage of $B_{\epsilon_1}(g^{j_1}(z_{j_1}))$ that contains $z_n \in B_\delta(z_{j_1})$, that is,

$$A_1 \subseteq g^{-j_1}(B_{\epsilon_1}(g^{j_1}(z_{j_1}))) \cap B_\delta(z_{j_1}).$$

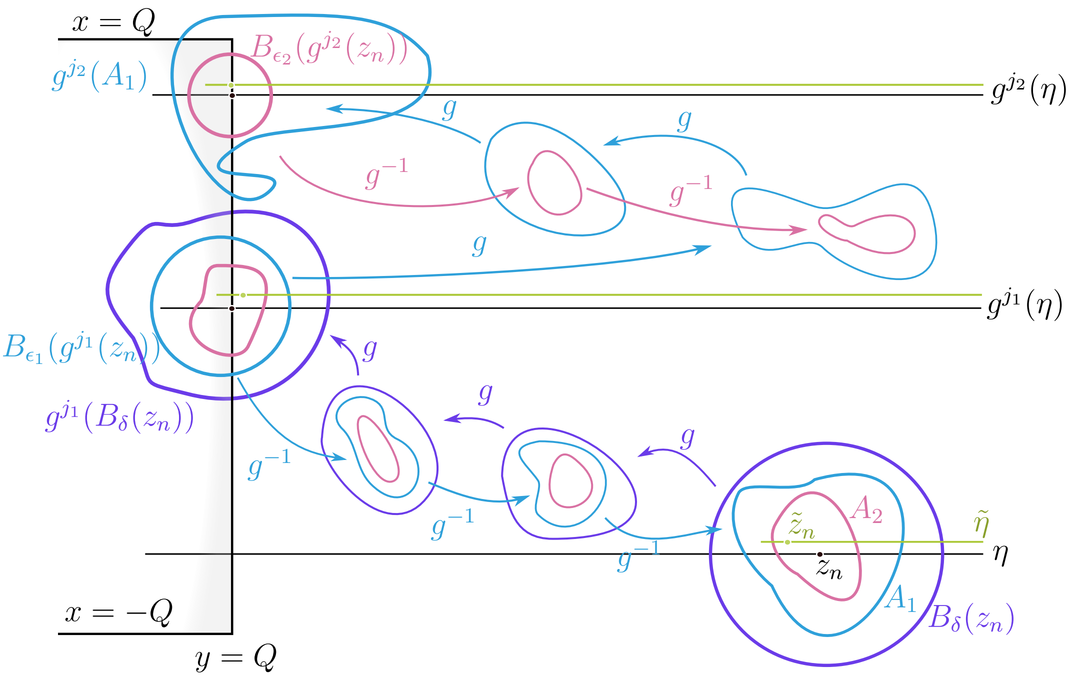

Figure 10: Construction of a neighbourhood of $z_n(\eta)$ in Claim 2 by pulling back balls centred at $g^j(z_n)$ for all $1 \leq j \leq n$ such that $g^j(z_n) \in \partial S_R$.

We claim that if $\tilde{\eta}$ is any other hair in $J(g)$ such that $\tilde{\eta} \cap A_1 \neq \emptyset$, then

$\tilde{z}_{j_1} \in A_1$. Indeed, by definition of A_1, the hair $g^{j_1}(\tilde{\eta})$ must intersect $B_{\epsilon_1}(g^{j_1}(z_{j_1}))$, and so, two cases can occur according to whether $g^{j_1}(\tilde{\eta})$ intersects S_R or not. If $g^{j_1}(\tilde{\eta}) \cap S_R = \emptyset$, by the assumption of $J(g)$ being a straight brush, the endpoint of the hair $g^{j_1}(\tilde{\eta})$ must be contained in $B_{\epsilon_1}(g^{j_1}(z_{j_1})) \setminus S_R$. Thus, A_1 contains the endpoint of $\tilde{\eta}$, that we denote by $\tilde{e}$, and so by the choice of δ, $f^j(\tilde{e}) \subset \mathbb{C} \setminus S_R$ for all $j < j_1$. Hence, $\tilde{z}_{j_1} = \tilde{e}$ is the endpoint of $\tilde{\eta}$. If on the other hand $g^{j_1}(\tilde{\eta}) \cap S_R \neq \emptyset$, since $B_{\epsilon_1}(g^{j_1}(z_{j_1}))$ is convex and $g^{j_1}(\tilde{\eta})$ is a horizontal straight line, the intersection $B_{\epsilon_1}(g^{j_1}(z_{j_1})) \cap g^{j_1}(\tilde{\eta}) \cap \partial S_R$ consists of a unique point, that we denote by p. In particular, by definition of $\tilde{z}_{j_1}$, $p \preceq f^{j_1}(\tilde{z}_{j_1})$, and so, if q is the preimage of p in A_1, $q \preceq \tilde{z}_{j_1}$ by Claim 1. But by the choice of δ, $f^j(q) \in \mathbb{C} \setminus S_R$ for all $1 \leq j < j_1$, and so, by minimality of $\tilde{z}_{j_1}$, it must occur that $\tilde{z}_{j_1} = q \in A_1$.

If $j_1 = n$ we have proved the claim. Otherwise, we can assume that ϵ_1 has been chosen small enough so that for all $j_1 < j \leq \min(j_2 - 1, n)$, $g^j(B_{\epsilon_1}(z_n)) \subset \mathbb{C} \setminus \overline{S}_R$. This implies that for any such j and any hair $\tilde{\eta}$ intersecting A_1, $g^j(\tilde{z}_{j_1}) \in \mathbb{C} \setminus \overline{S}_R$, and thus, $\tilde{z}_{j_1} = \ldots = \tilde{z}_{\min(j_2-1,n)}$.

If $k = 1$ in (4.3.5) we are done. Otherwise, we aim to define a neighbourhood $A_2 \subset A_1$ of $z_{j_2}(= z_n)$ with analogous properties to those of A_1. That is, such that if $\tilde{\eta} \cap A_2 \neq \emptyset$ for some hair $\tilde{\eta}$, then $\tilde{z}_{j_2} \in A_2$. In order to do so, by the same argument as when we chose ϵ_1, we can choose ϵ_2 such that

$$B_{\epsilon_2}(g^{j_2}(z_{j_2})) \subset g^{j_2}(A_1) \quad \text{ and } \quad |Q - \operatorname{Im}(g^{j_2}(z_{j_2}))| < \epsilon_2.$$

Let A_2 be the connected component of $g^{-j_2}(B_{\epsilon_2}(g^{j_2}(z_n))) \cap A_1$ that contains z_n, and let $\tilde{\eta}$ be any other hair in $J(g)$ such that $\tilde{\eta} \cap A_2 \neq \emptyset$. By definition, since $g^{j_2}(\tilde{\eta})$ is a horizontal straight line, if $g^{j_2}(\tilde{z}_{j_2-1}) \in \mathbb{C} \setminus S_R$, then $\tilde{z}_{j_2-1} = \tilde{z}_{j_2} \in A_2$. Otherwise, $g^{j_2}(\tilde{\eta})$ intersects ∂S_R in a single point $p \in B_{\epsilon_2}(g^{j_2}(z_n))$ such that $p \succeq g^{j_2}(\tilde{z}_{j_2-1})$, and so $\tilde{z}_{j_2}$ is the preimage of p in $A_2 \subset A_1$. If $j_2 \neq n$, then choose ϵ_2 small enough so that $g^j(B_{\epsilon_2}(z_n)) \subset \mathbb{C} \setminus \overline{S}_R$ for all $j_2 \leq j < \min(j_3 - 1, n)$ and hence, for any $\tilde{\eta}$ intersecting A_2 and all $j_1 < j \leq \min(j_3 - 1, n)$, $\tilde{z}_{j_2} = \ldots = \tilde{z}_{\min(j_3-1,n)}$.

Continuing the process for each j_i in (4.3.5), we build a nested sequence of open sets $A_k \subseteq \cdots \subseteq A_i \subset \cdots \subseteq A_1$ such that $\tilde{z}_n \in A_k \subset B_\delta(z_n)$ for any hair $\tilde{\eta}$ such that $\tilde{\eta} \cap A_k \neq \emptyset$. Hence, choosing any α so that $B_\alpha(z_n) \subset A_k$, the claim follows. $\triangle$

Continuity of the functions $\{\pi_n\}_{n \in \mathbb{N}}$ is now a consequence of Claim 2: let us fix $n \in \mathbb{N}$, let $z \in J(g)$, in particular belonging to some hair η, and fix any $\epsilon > 0$. We want to see that there exists $\delta > 0$ such that $\pi_n(B_\delta(z)) \subset B_\epsilon(\pi_n(z))$. Three cases might occur:

- $z \succ z_n(\eta)$, and so $\pi_n(z) = z$. In particular, $g^j(z) \in \mathbb{C} \setminus \overline{S}_R$ for all $j \leq n$, and thus, we can choose $\delta < \epsilon$ small enough such that $g^j(B_\delta(z)) \subset \mathbb{C} \setminus \overline{S}_R$ for all $0 \leq j \leq n$. This already implies that $\pi_n(w) = w$ for all $w \in B_\delta(z) \subset B_\epsilon(z)$, since by Proposition 4.26, if $\tilde{\eta}$ is the hair containing w, then $w \prec z_n(\tilde{\eta})$ only if $g^j(w) \in S_R$ for some $j \leq n$, which cannot occur by the choice of δ.

- $z \prec z_n(\eta)$, and hence $\pi_n(z) = z_n(\eta)$ and z is not the endpoint of η. Let $\alpha := \alpha_n(\eta)$ be the constant given by Claim 2. If we choose $\delta < \min(\alpha, \epsilon)$, by $J(g)$ being a straight brush, any hair intersecting $B_\delta(z)$ also intersects $B_\alpha(z_n(\eta))$. By Claim 2, if $w \in \tilde{\eta} \cap B_\delta(z)$ for some hair $\tilde{\eta}$ and $w \prec z_n(\tilde{\eta})$, then $\pi_n(w) = z_n(\tilde{\eta}) \in B_{\min(\alpha,\epsilon)}(z_n(\eta))$. Otherwise, the case of $w \succeq z_n(\tilde{\eta})$, and so $\pi_n(w) = w$, can only occur when $w \in B_\delta(z) \cap B_{\min(\alpha,\epsilon)}(z_n(\eta))$, and thus again $\pi_n(w) \in B_\epsilon(z_n(\eta))$, as we wanted to show.

- $z = z_n(\eta)$, and hence $\pi_n(z) = z$. If $g^j(z_n(\eta)) \in \mathbb{C} \setminus \overline{S}_R$ for all $j \leq n$, then z is the endpoint of η and the same argument as in the first case applies. If on the contrary $g^j(z_n(\eta)) \in \partial S_R$ for some $j \leq n$, then the same argument as in the second case applies.

Now that we have shown that the functions $\{\pi_n\}_{n\in\mathbb{N}}$ are continuous, our next goal is to prove that they converge to a limit function. In order to do so, we start by showing that for each hair η of $J(g)$, the sequence $\{z_n(\eta)\}_{n\in\mathbb{N}}$ is convergent by being a Cauchy sequence. By Claim 1, for each $n \geq 1$, either $z_{n-1} = z_n$ or $z_{n-1} \prec z_n$, the latter case occurring only if $g^n(z_{n-1}) \in S_R \cap \{z : 0 \leq \operatorname{Re}(z) < Q\}$ and $g^n(z_n) \in \partial S_R$. Thus, in the latter case, the Euclidean length of the piece of the hair $g^n(\eta)$ joining $g^n(z_{n-1})$ and $g^n(z_n)$, that we denote by γ, is at most Q. That is,

$$\ell_{\text{eucl}}(\gamma) \leq Q. \tag{4.3.6}$$

Since the map g is of disjoint type, and in particular hyperbolic, we can find an open neighbourhood U of $P(g)$ such that $g(U) \subset \overline{U}$, see Proposition 2.20. We might assume without loss of generality that U has finite Euclidean perimeter and a smooth boundary, since otherwise we can take a slightly smaller domain $P(g) \subset U' \subset U$ with such properties. Let $W := \mathbb{C} \setminus \overline{U}$ and define the set of tracts $\mathcal{T}_g$ as the connected components of $g^{-1}(W)$, that satisfies $\mathcal{T}_g \subseteq g^{-1}(W) \subset W$.

Thus, $g : \mathcal{T}_g \to W$ is a covering map, and

$$J(g) = \bigcap_{n=1}^{\infty} g^{-n}(W) \Subset W. \tag{4.3.7}$$

We can endow W with the hyperbolic metric induced from its universal covering map. Then, g expands uniformly the hyperbolic metric of W, that is, there exists a constant $\Lambda > 1$ such that $||Dg(z)||_W \geq \Lambda$ for all $z \in \mathcal{T}_g$, see Proposition 2.25.

By Proposition 2.19(4), only finitely many pieces of tracts in $\mathcal{T}_g$ intersect $\overline{S}_R$. Let us consider the collection

$$\{K_1, \dots, K_{\tilde{M}}\} \tag{4.3.8}$$

of the closures of such pieces of tracts. By the choice of ∂U being smooth and analytic, so are the boundaries of the tracts in $\mathcal{T}_g$, and in particular the boundaries of the sets $\{K_i\}_{i=1}^{\tilde{M}}$. By this and since $K_i \Subset W$ for each $i \leq \tilde{M}$, the density function ρ_W is continuous in each compact set $\overline{K}_i$, and so, by (4.3.7), it attains a maximum value M_i on it. Let $M \coloneqq \max_i M_i$. Recall that the straight line γ joining $g^n(z_{n-1})$ and $g^n(z_n)$ belongs to $J(g) \cap \overline{S}_R$, since in particular is a piece of hair. Hence, by (4.3.7), γ must be totally contained in one of the compact sets $\{K_i\}_i$, and by (4.3.6), we have the following bound for its hyperbolic length:

$$\ell_W(\gamma) = \int |\gamma'(t)| \rho_W(\gamma(t)) dt \leq M \int |\gamma'(t)| dt = M \cdot \ell_{\text{eucl}}(\gamma) \leq M \cdot Q. \tag{4.3.9}$$

Let β be the piece of the hair η joining z_{n-1} and z_n. Then, by (4.3.7) and since $||Dg(z)||_W \geq \Lambda$ for all $z \in \beta$, by the same argument as that in the proof of Corollary 3.13,

$$d_W(z_{n-1}, z_n) \leq \ell_W(\beta) \leq \frac{\ell_W(\gamma)}{\Lambda^n} \leq \frac{M \cdot Q}{\Lambda^n}. \tag{4.3.10}$$

Note that the upper bounds on the lengths of γ and β do not depend on the points z_{n-1} and z_n but only on them belonging to S_R, and hence $\{z_n(\eta)\}_{n\in\mathbb{N}}$ forms a Cauchy sequence in the complete space (W, ρ_W), that converges to a limit point, that we denote by z_η. Consequently, we have shown that the functions $\{\pi_n\}_{n\in\mathbb{N}}$ converge to a continuous limit function π such that

$$\text{if } z \in \eta, \quad \text{then} \quad \pi(z) = \begin{cases} z_\eta & \text{if } z \prec z_\eta, \\ z & \text{if } z \succeq z_\eta. \end{cases} \tag{4.3.11}$$

In particular, using the definition of the functions π_n in (4.3.4), the limit function

π must be equal to that defined in (4.3.3), and thus we have proved the first part of the statement.

For the second part of the statement we have the additional assumption that g has a logarithmic transform $G : \mathcal{T}_G \to \mathbb{H}_{\log L}$ such that $\mathcal{T}_G \subset \mathbb{H}_{\log L+8\pi}$, and we aim to get for each $z \in J(g)$ an estimate on the Euclidean distance between z and $\pi(z)$. Note that this assumption on g implies that $J(g) \subset \mathbb{C} \setminus \mathbb{D}_{L\,\mathrm{e}^{8\pi}}$. Fix any $z \in J(g)$ and let η be the hair it belongs to. If $z \succeq z_\eta$ then $\pi(z) = z$. Otherwise, by (4.3.11), $\pi(z) = z_\eta$ and z_η cannot be the endpoint of η. Thus, it must occur that $g^n(z_\eta) \in \partial S_R$ for some $n \geq 0$, and in particular,

$$g^n(z) \in (S_R \setminus \mathbb{D}_{L\,\mathrm{e}^{8\pi}}) \cap \{z : 0 \leq \operatorname{Re}(z) < Q\} \tag{4.3.12}$$

by $J(g)$ being a straight brush. Let w and w_η be a pair of respective preimages of z and z_η under the exponential map, lying in the same connected component of $J(G)$. In particular, each hair of $J(g)$ must be lifted to a connected component of $J(G)$, and thus $w, w_\eta \in J_{\underline{s}}(G)$ for some $\underline{s} \in \operatorname{Addr}(J(G))$. Moreover, recall from (4.3.8) that only finitely many pieces of tracts $\mathcal{T}_g$ intersect S_R, and hence, $g^n(z)$ and $g^n(z_\mu)$ must belong to one of the compact sets in (4.3.8). Hence, only finitely many of the *different* logarithmic tracts $\mathcal{T}_G$ of G, that is, up to their $2\pi i$-translates, intersect $\exp^{-1}(S_R)$, and all of them lie in the vertical strip $V(\log L + 8\pi, Q)$. Since each of these pieces of logarithmic tracts must be disjoint from their $2\pi i$-translates, there exists an upper bound for the Euclidean diameter of any of them, say $\varDelta > 0$. Thus, by (2.5.1), $G^n(w)$ and $G^n(w_\eta)$ belong to the same one of these pieces of tracts, and hence $|G^n(w) - G^n(w_\eta)| \leq \varDelta$. By this and Proposition 2.37,

$$|w - w_\eta| \leq \frac{1}{2^n}|G^n(w) - G^n(w_\eta)| \leq \frac{\varDelta}{2^n} < \varDelta. \tag{4.3.13}$$

In particular, $w_\eta \in \overline{V(\operatorname{Re} w - \varDelta, \operatorname{Re} w + \varDelta)}$, and so, if we denote $M := \exp(\varDelta)$, then $z_\eta \in \overline{A(M^{-1}|z|, M|z|)}$. □

Observation 4.32 (Action of the map π_R)**.** Following Theorem 4.31, let η be a hair of $J(g)$. Then, by Proposition 4.27, there exists another hair $\tilde{\eta}$ such that g maps η bijectively to $\tilde{\eta}$. This together with the definition of $z_{\tilde{\eta}}$ implies that either both $z_\eta, z_{\tilde{\eta}} \in \mathbb{C} \setminus S_R$ and then $g(z_\eta) = z_{\tilde{\eta}}$, or $z_\eta \in \partial S_R$ and $z_{\tilde{\eta}} \prec g(z_\eta)$. In the latter case, if x is the preimage of $z_{\tilde{\eta}}$ in η, then by Proposition 4.26, $x \in S_R$. In particular, we have shown that in any case $g(\pi_R(\eta)) \subseteq \pi_R(\tilde{\eta})$.

We conclude this section by pointing out the close relation between disjoint type functions that have a Julia Cantor bouquet, and disjoint type criniferous functions. It follows from Corollary 4.28 that all disjoint type functions with Julia Cantor bouquets are criniferous. In order to study if the converse holds, it is more convenient for us to use the following characterization of Cantor bouquets.

Theorem 4.33 (Characterization of Cantor bouquets [ARG17, Theorem 2.8]). *A set $X \subset \mathbb{C}$ is a Cantor bouquet if and only if the following conditions are satisfied:*

(1) X is closed.

(2) Every connected component of X is an arc connecting a finite endpoint to infinity.

(3) For any sequence y_n converging to a point y, the arcs $[y_n, \infty)$ converge to $[y, \infty)$ in the Hausdorff metric.

(4) The endpoints of X are dense in X.

(5) If $x \in X$ is accessible from $\mathbb{C}\setminus X$, then x is an endpoint of X. (Equivalently, every hair of X is accumulated on by other hairs from both sides.)

Proposition 4.34 (Properties of disjoint type criniferous functions). *If f is a criniferous disjoint type function, then conditions (1), (2), (4) and (5) in Theorem 4.33 are satisfied for $J(f) \subset \mathbb{C}$.*

Proof. Item (1) holds by definition of Julia set, and (2) follows from [RG16, Corollary 6.6] and [BRG17, Remark 4.15]. By Theorem 2.30 together with Observation 2.22, each Julia constituent of f is either a ray tail or a dynamic ray together with its endpoint. Hence, all repelling periodic points and points with bounded orbits are endpoints of these curves. Since repelling periodic points are dense in $J(f)$, item (4) follows. By [RG16, Theorem 2.3], any point accessible from $F(f)$ must be an endpoint of a connected component of $J(f)$, and this proves item (5). □

Remark. It follows from the previous proposition that for a disjoint type function, having a Julia Cantor bouquet would be equivalent to being criniferous as long as (3) in Proposition 4.34 holds for all disjoint type criniferous functions. However, whether this is the case or not remains an **open question**.

4.4 The class $\mathcal{CB}$

This section is devoted to the definition and basic properties of the class $\mathcal{CB}$ of functions. Recall from Chapter 1 that

$$\mathcal{CB} := \left\{ \begin{array}{c} f \in \mathcal{B} : \text{ exists } \lambda \in \mathbb{C} : g_\lambda := \lambda f \text{ is of disjoint type} \\ \text{and } J(g_\lambda) \text{ is a Cantor bouquet} \end{array} \right\}.$$

In particular, we study some properties that all functions belonging to this class share and we prove Theorem 1.7. Recall that two maps $f, g \in \mathcal{B}$ are said to be *quasiconformally equivalent ($\sim$ near infinity)* if there exist quasiconformal maps $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ such that

$$\psi(g(z)) = f(\varphi(z)) \tag{4.4.1}$$

for all $z \in \mathbb{C}$ ($\sim$ whenever $|f(z)|$ or $|g(\varphi(z))|$ is large enough). Moreover, for an entire function with bounded singular set, its *parameter space* is the collection of all entire functions quasiconformally equivalent to it.

Proposition 4.35 (All functions in $\mathcal{B}$ have disjoint type maps in their parameter space)**.** *Let $f \in \mathcal{B}$. Then, for all $\lambda \in \mathbb{C}^*$ with small enough modulus, the maps $h_\lambda := \lambda f$ and g_λ given by $g_\lambda(z) := f(\lambda z)$ are of disjoint type and belong to the parameter space of f.*

Proof. The maps h_λ and g_λ are trivially quasiconformally equivalent to f; (4.4.1) holds for g_λ by taking ψ to be the identity map and φ as $z \mapsto \lambda z$, and for h_λ, (4.4.1) holds taking ψ to be $z \mapsto z/\lambda$ and φ the identity map. Since $f \in \mathcal{B}$, we can choose $R > 0$ such that $\{S(f), 0, f(0)\} \subset \mathbb{D}_R$. For $\lambda \in \mathbb{C}$ with $|\lambda|$ sufficiently small, $f(\lambda \mathbb{D}_R) \subset \mathbb{D}_R$ and $\lambda f(\mathbb{D}_R) \subset \mathbb{D}_R$. Thus, for any such λ, $g_\lambda(\mathbb{D}_R) \subset \mathbb{D}_R$ and $h_\lambda(\mathbb{D}_R) \subset \mathbb{D}_R$. Moreover, for every $\lambda \in \mathbb{C}^*$, it is easy to see that $S(h_\lambda) = \lambda S(f) \subset \mathbb{D}_R$ and $S(g_\lambda) = S(f) \subset \mathbb{D}_R$. The combination of these two facts is by Proposition 2.20 enough to characterize g_λ and h_λ as disjoint type maps. □

Note that if two maps are quasiconformally equivalent to a third one, then they are quasiconformally equivalent to each other, since the composition of two quasiconformal maps is quasiconformal. In particular, all disjoint type maps in the parameter space of a function $f \in \mathcal{B}$ are pairwise quasiconformally equivalent, and the following result tells us that their dynamics are closely related.

Theorem 4.36 (Conjugacy between disjoint type maps [Rem09, Theorem 3.1])**.** *Any two quasiconformally equivalent disjoint type maps are conjugate on their Julia sets.*

In particular, a conjugacy between Julia sets implies that the topological structure of the sets must be the same. Even if in general this does not necessarily imply that their embeddings in the plane are the same (i.e. they might not be ambiently homeomorphic), it follows from the proof of Theorem 4.36 that the map that conjugates two functions as in Theorem 4.36 is an ambient homeomorphism.

Corollary 4.37. *Let $f \in \mathcal{B}$. Then any two disjoint type maps on its parameter space are conjugate on their Julia sets, and hence these sets are homeomorphic.*

This corollary implies that in order to determine if a function $f \in \mathcal{B}$ belongs to $\mathcal{CB}$, it is enough to check the topological structure of the Julia set of any disjoint type map quasiconformally equivalent to it:

Proposition 4.38 (Class $\mathcal{CB}$)**.** *A transcendental entire function f belongs to $\mathcal{CB}$ if and only if the Julia set of any (and thus all) disjoint type function on its parameter space is a Cantor bouquet.*

We note that even if not under this terminology, many functions in $\mathcal{CB}$ have already been studied before, and in particular, all maps in both the exponential and cosine families belong to it, see Proposition 4.39(A). To illustrate this, we gather together, up to date and to the author's knowledge, classes of functions appearing on the literature that are known to be in $\mathcal{CB}$:

Proposition 4.39 (Sufficient conditions for functions in $\mathcal{CB}$)**.** *A transcendental entire function $f \in \mathcal{B}$ is in class $\mathcal{CB}$ if one of the following holds:*

(A) f is a finite composition of functions of finite order,

(B) there exists a logarithmic transform F of f that satisfies a linear head-start condition for some φ (see Definition 2.38),

(C) there exists a logarithmic transform G of a disjoint type function g given by $g(z) := f(\lambda z)$ for $\lambda \in \mathbb{C}^*$ small enough, such that G satisfies a uniform head-start condition for some φ.

Proof. If $f \in \mathcal{B}$ is a finite composition of functions of finite order, then by [RRRS11, Theorem 5.6 and Lemma 5.7], there exists a logarithmic transform $F\colon \mathcal{T}_F \to \mathbb{H}_{\log L}$ of f satisfying a linear head-start condition for some φ. In this case, the map

$$G : \mathcal{T}_G = (\mathcal{T}_F - \log \lambda) \to \mathbb{H}_{\log L} \quad \text{given by} \quad G(w) := F(w + \log \lambda)$$

is a logarithmic transform of the map g given by $g(z) := f(\lambda z)$, since in that case $\mathrm{e}^{F(w+\log\lambda)} = f(\mathrm{e}^{w+\log\lambda}) = f(\lambda\,\mathrm{e}^{w}) = g(\mathrm{e}^{w}) = \mathrm{e}^{G(w)}$. Moreover, since F satisfies a linear head-start condition, by its definition, G also satisfies a linear (in particular uniform) head-start condition, and for λ sufficiently small, by Propositions 4.35 and 2.20, G and g are of disjoint type. By [BJR12, Corollary 6.3], since G satisfies a uniform head-start condition and is of disjoint type, both $J(G)$ and $J(g)$ are Cantor bouquets. We have shown that $(A) \Rightarrow (B) \Rightarrow (C) \Rightarrow f \in \mathcal{CB}$, and so the statement follows. $\square$

Our next goal is to show that if a function belongs to class $\mathcal{CB}$, then all its iterates also belong to $\mathcal{CB}$. This will be a consequence of the following property on composition of maps that are quasiconformally equivalent near infinity. We use in our proof some tools from quasiconformal maps gathered in [Rem09, Section 2]. We refer to [LV73, Vuo88] for definitions. In particular, we will make use of the following auxiliary result regarding interpolation of quasiconformal maps on annuli. The original source is [Leh65], but we present in the next proposition the reformulation given in [Rem09, Proposition 2.11].

Proposition 4.40 (Interpolation of quasiconformal maps on annuli [Leh65])**.** *Let $A, B \subset \mathbb{C}$ be two bounded annuli, each bounded by two Jordan curves. Suppose that $\psi, \varphi \colon \mathbb{C} \to \mathbb{C}$ are quasiconformal maps such that ψ maps the inner boundary α^- of A to the inner boundary β^- of B, and φ takes the outer boundary α^+ of A to the outer boundary β^+ of B. Then there is a quasiconformal map $\tilde{\varphi} \colon \mathbb{C} \to \mathbb{C}$ that agrees with ψ on the bounded component of $\mathbb{C} \setminus A$ and with φ on the unbounded component of $\mathbb{C} \setminus A$.*

Proposition 4.41 (Composition of quasiconformally equivalent maps)**.** *Suppose that $f_1, g_1 \in \mathcal{B}$ are quasiconformally equivalent near infinity to f_2 and g_2, respectively. Then, $f_1 \circ g_1$ is quasiconformally equivalent near infinity to $f_2 \circ g_2$.*

Proof. By assumption, there exist quasiconformal maps $\varphi_g, \varphi_g, \psi_g, \psi_f : \mathbb{C} \to \mathbb{C}$ such that

$$\psi_f(f_1(z)) = f_2(\varphi_f(z)) \qquad \text{and} \qquad \psi_g(g_1(z)) = g_2(\varphi_g(z)) \tag{4.4.2}$$

whenever $\max\{|f_1(z)|, |f_2(\varphi_f(z))|\} > R_f$ and $\max\{|g_1(z)|, |g_2(\varphi_g(z))|\} > R_g$ for some fixed $R_f, R_g > 0$. Equivalently, the semiconjugacies between the functions f_1 and f_2, and g_1 and g_2, are respectively defined in the sets

$$\begin{aligned} A(R_f) &:= f_1^{-1}(\mathbb{C} \setminus \mathbb{D}_{R_f}) \cup \varphi_f^{-1}(f_2^{-1}(\mathbb{C} \setminus \mathbb{D}_{R_f})) \quad \text{and} \\ B(R_g) &:= g_1^{-1}(\mathbb{C} \setminus \mathbb{D}_{R_g}) \cup \varphi_g^{-1}(g_2^{-1}(\mathbb{C} \setminus \mathbb{D}_{R_g})). \end{aligned} \tag{4.4.3}$$

By increasing the constant R_f, we can assume that $A(R_f) \Subset \mathbb{C}\setminus(\mathbb{D}_{R_g} \cup \psi_g^{-1}(\mathbb{D}_{R_g}))$ and that there exists an annulus $\mathcal{R} \subset \mathbb{C}$ such that $\mathbb{D}_{R_g} \cup \psi_g^{-1}(\mathbb{D}_{R_g})$ is compactly contained in the bounded component of $\mathbb{C}\setminus\overline{\mathcal{R}}$, $A(R_f)$ in the unbounded one, and such that if $\mathcal{R}^-$ and $\mathcal{R}^+$ are the inner an outer boundaries of $\mathcal{R}$, then the curves $\psi_g(\mathcal{R}^-)$ and $\varphi_f(\mathcal{R}^+)$ are the respective inner an outer boundaries of a topological annulus. See Figure 11.

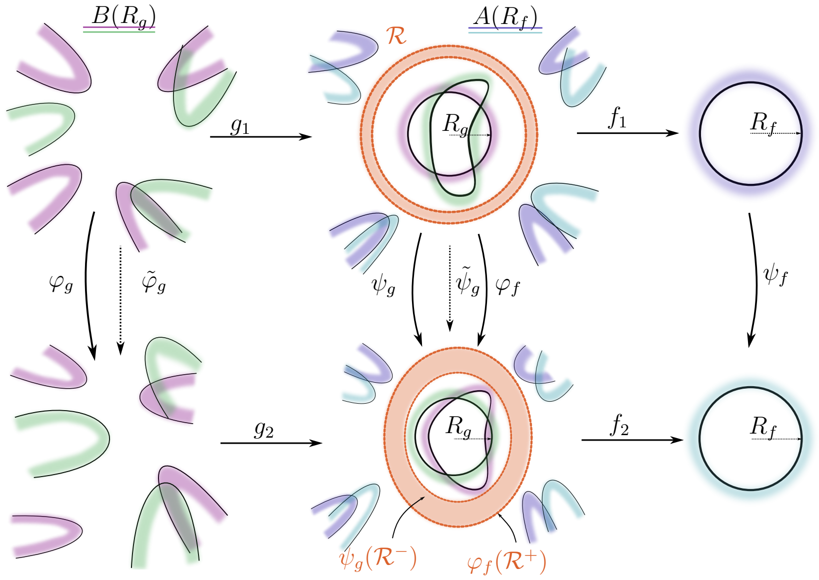

Figure 11: Proof of Proposition 4.41 by interpolating the maps ψ_g and φ_f using the annulus shown in orange.

By Proposition 4.40, we can interpolate ψ_g and φ_f the following way: there exists a quasiconformal map $\tilde{\psi}_g : \mathbb{C} \to \mathbb{C}$ that agrees with ψ_g in the bounded component of $\mathbb{C}\setminus\mathcal{R}$ and with φ_f in the unbounded component of $\mathbb{C}\setminus\mathcal{R}$. In particular, since by construction $\tilde{\psi}_g \equiv \psi_g$ in $(\partial\mathbb{D}_{R_g} \cup \psi_g^{-1}(\mathbb{D}_{R_g}))$ and by the Alexander trick the isotopy class of a homeomorphism between two Jordan domains is determined by its boundary values,

$$\tilde{\psi}_g \text{ is isotopic to } \psi_g \text{ in } X := \mathbb{C} \setminus (\mathbb{D}_{R_g} \cup \psi_g^{-1}(\mathbb{D}_{R_g})) \text{ relative to } \partial X. \tag{4.4.4}$$

Our next goal is to replace the map φ_g by another map $\tilde{\varphi}_g$ by "*lifting*" the map $\tilde{\psi}_g$ in such a way that the relation $\tilde{\psi}_g(g_1(z)) = g_2(\tilde{\varphi}_g(z))$ holds for all $z \in B(R_g)$. In order to do so, note that by construction, the restriction of φ_g to any connected component $C \subset B(R_g)$ is a homeomorphism into its image, and moreover, the closure of each domain $\varphi_g(C)$ is by definition mapped injectively to $\mathbb{C} \setminus (\mathbb{D}_{R_g} \cup \psi_g(\mathbb{D}_{R_g}))$ under g_2. Hence, the inverse branches of g_2 from $\mathbb{C} \setminus (\mathbb{D}_{R_g} \cup \psi_g(\mathbb{D}_{R_g}))$ to each $\varphi_g(C)$, which we denote by $g_2^{-1}|_{\varphi_g(C)}$, are well-defined, and by (4.4.2),

$\varphi_g(C) = (g_2^{-1}|_{\varphi_g(C)} \circ \psi_g \circ g_1)(C)$. We define from each connected component C of $B(R_g)$ a map $\tilde{\varphi}_g|_C : C \to \varphi_g(C)$ as

$$\tilde{\varphi}_g|_C := g_2^{-1}|_{\varphi_g(C)} \circ \tilde{\psi}_g \circ g_1.$$

By (4.4.4) and since $\tilde{\psi}_g \equiv \psi_g$ in $g_1(\partial C)$, we have that φ_g and $\tilde{\varphi}_g|_C$ are isotopic in C, and moreover, by construction, the continuous extension of $\tilde{\varphi}_g|_C$ to ∂C equals $\varphi_g|_{\partial C}$. Thus, since connected components are pairwise disjoint, we can then define a homeomorphism of the plane $\tilde{\varphi}_g : \mathbb{C} \to \mathbb{C}$ as

$$\tilde{\varphi}_g(z) := \begin{cases} \tilde{\varphi}_g|_C & \text{if } \quad z \in C \text{ for some } i \\ \varphi_g(z) & \text{otherwise.} \end{cases}$$

By the glueing lemma, see [Rem09, Proposition 2.10], $\tilde{\varphi}_g$ is a quasiconformal map. Consequently, $\tilde{\psi}_g(g_1(z)) = g_2(\tilde{\varphi}_g(z))$ for all $z \in B(R_g)$. Since by construction $\varphi_f \equiv \tilde{\psi}_g$ in $A(R_f)$, we have that

$$(\psi_f \circ f_1 \circ g_1)(z) = (f_2 \circ \varphi_f \circ g_1)(z) = (f_2 \circ \tilde{\psi}_g \circ g_1)(z) = (f_2 \circ g_2 \circ \tilde{\varphi}_g)(z)$$

for all $z \in \mathbb{C}$ such that $\max\{|f_1(g_1(z))|, |f_2(g_2(\tilde{\varphi}_g(z))|\} > R_f$, and so we have proved the proposition. $\square$

Corollary 4.42 (Class $\mathcal{CB}$ is closed under iteration)**.** *If $f \in \mathcal{CB}$, then $f^n \in \mathcal{CB}$ for all $n \geq 0$.*

Proof. By Proposition 4.35, we can choose $\lambda \in \mathbb{C}^*$ such that $h_\lambda := \lambda f$ is a disjoint type function quasiconformally equivalent to f. Let us fix any $n > 0$. By Proposition 4.41 used recursively $n-1$ times, the function f^n is quasiconformally equivalent near infinity to h_λ^n. Moreover, again by Proposition 4.35, we can choose $\mu \in \mathbb{C}^*$ such that $g_\mu := \mu f^n$ is a disjoint type function quasiconformally equivalent to f^n. Then, g_μ and h_λ^n are two disjoint type functions quasiconformally equivalent near infinity, and hence they are quasiconformally conjugate on a neighbourhood of infinity [Rem09, Theorem 1.4]. Moreover, by taking μ and λ with sufficiently small modulus, one can see that that the conjugacy from [Rem09] extends to a neighbourhood of their Julia sets. In particular, by assumption $J(h_\lambda)$ is a Cantor bouquet, and since Julia sets are completely invariant, $J(h_\lambda^n)$, and thus $J(g_\mu)$, are also Cantor bouquets. Consequently, we have shown that $f^n \in \mathcal{CB}$. $\square$

Conjugacy near infinity

In order to complete the proof of Theorem 1.7, we are left to show that all functions in $\mathcal{CB}$ are criniferous. This will be a consequence of a more general result from [Rem09], that can be regarded as a sort of analogue of Böttcher's Theorem for functions in class $\mathcal{B}$. More specifically, that result (see Corollary 4.45) allows us to conjugate the dynamics *near infinity* of any function $f \in \mathcal{CB}$ to those of a disjoint type function in its parameter space. In particular, since by assumption any such disjoint type map has a Julia Cantor bouquet, this semiconjugacy will imply the existence of ray tails for f, and subsequently Theorem 1.7. We remark that for us, the importance of this section goes beyond Theorem 1.7, as we set here the ground to prove Theorem 1.8: Corollary 4.45 together with Proposition 4.49 will provide a first approximation of the semiconjugacy we aim to build between the model space and $J(f)$.

4.43. Let us fix $f \in \mathcal{CB}$ and choose any $K > 0$ such that $S(f) \subset \mathbb{D}_K$. Let $L \geq K$ be any constant sufficiently large such that $f(\mathbb{D}_K) \subset \mathbb{D}_L$. In particular, no preimage of $\mathbb{C} \setminus \mathbb{D}_L$ intersects $\mathbb{D}_K$, that is,

$$f^{-1}(\mathbb{C} \setminus \mathbb{D}_L) \subset \mathbb{C} \setminus \mathbb{D}_K. \tag{4.4.5}$$

Denote

$$\lambda := \frac{K}{e^{8\pi}L} \quad \text{and define} \quad g := g_\lambda \colon \mathbb{C} \mapsto \mathbb{C} \quad \text{as} \quad g_\lambda(z) := f(\lambda z). \tag{4.4.6}$$

Then, for each $z \in \mathbb{C}$ such that $|g(z)| = |f(\lambda z)| > L$, by (4.4.5), $|\lambda z| > K$ and hence, $|z| > e^{8\pi}L$. That is,

$$g^{-1}(\mathbb{C} \setminus \mathbb{D}_L) \subset \mathbb{C} \setminus \mathbb{D}_{e^{8\pi}L}. \tag{4.4.7}$$

Consequently, by Propositions 2.20 and 4.35, g is a disjoint type map in the parameter space of f. Let $\mathcal{T}_f$ and $\mathcal{T}_g$ be the set of tracts of f and g defined as the respective connected components of $f^{-1}(\mathbb{C} \setminus \mathbb{D}_L)$ and $g^{-1}(\mathbb{C} \setminus \mathbb{D}_L)$. Next, we fix logarithmic transforms for f and g: let

$$F : \mathcal{T}_F \to \mathbb{H}_{\log L} \text{ with } \mathcal{T}_F := \exp^{-1}(\mathcal{T}_f) \quad \text{and} \quad \mathbb{H}_{\log L} := \exp^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_L}),$$

which by (4.4.5) satisfies $\mathcal{T}_F \subset \mathbb{H}_{\log K}$. Then, the map

$$G := G_\lambda : \mathcal{T}_G = (\mathcal{T}_F - \log \lambda) \to \mathbb{H}_{\log L} \quad \text{given by} \quad G(w) := F(w + \log \lambda)$$

is a logarithmic transform for g, since $\mathrm{e}^{F(w+\log\lambda)} = f(\mathrm{e}^{w+\log\lambda}) = f(\lambda\,\mathrm{e}^{w}) = g(\mathrm{e}^{w}) = \mathrm{e}^{G(w)}$. By the choice of the constant λ in (4.4.6), it holds that

$$\mathcal{T}_G \subset \mathbb{H}_{\log L+8\pi}. \tag{4.4.8}$$

We can now state the aforementioned result from [Rem09] for the logarithmic transforms F and G just defined. In order to do so, for any logarithmic transform F and constant $Q>0$ we denote

$$J_Q(F) := \{z \in J(F) : \operatorname{Re}(F^n(z)) \geq Q \text{ for all } n \geq 1\}.$$

The following result is a compendium of [Rem09, Theorem 3.2, Lemma 3.3 and Theorem 3.4], in a version adapted to our setting[10]:

Theorem 4.44 (Conjugacy near infinity for logarithmic transforms)**.** *Let λ be the constant and let F and G be the logarithmic transforms fixed in 4.43. For every constant $Q > 2|\log\lambda|+1$, there is a continuous map*[11] *$\Theta := \Theta^{\lambda}\colon J_Q(G) \to J(F)$ such that*

$$\Theta \circ G = F \circ \Theta, \qquad |\Theta(w) - w| \leq 2|\log\lambda| \tag{4.4.9}$$

and is a homeomorphism onto its image. Moreover, $J_{2Q}(F) \subset \Theta(J_Q(G))$ and Θ can be chosen so that $\Theta(w+2\pi i) = \Theta(w) + 2\pi i$.

Remark. In [Rem09, Theorem 3.2], it is assumed that the logarithmic transforms F and G are *normalized*. Even if this might not be the case for the transforms F and G fixed in 4.43, all that is required in the proof of [Rem09, Theorem 3.2] is that G satisfies the inclusion in (4.4.7), and thus it applies to them.

We can transfer this result to the dynamical planes of f and g. Recall from (4.3.1) that for each entire function g and constant $R \geq 0$, we denote

$$J_R(g) := \{z \in J(g) : |g^n(z)| \geq R \text{ for all } n \geq 1\} \quad \text{and} \quad I_R(g) := I(g) \cap J_R(g).$$

Theorem 4.44 has the following implications:

Corollary 4.45 (Conjugacy near infinity in the dynamical plane)**.** *Let λ be the constant and let f and g be the functions fixed in 4.43. Then, for every constant $R > \exp(2|\log\lambda| + 8\pi + \log L)$, there exists a continuous map $\theta := \theta_R\colon J_R(g) \to J(f) \cap \mathcal{T}_f$ such that*

$$\theta \circ g = f \circ \theta \tag{4.4.10}$$

[10] [Rem09, page 250] with $F_0 = G$, $\kappa = -\log\lambda$, and $F_\kappa = F$.

[11] The map Θ extends to a quasiconformal map $\Theta : \mathbb{C} \to \mathbb{C}$. See [Rem09, Theorem 3.4 or Theorem 1.1].

and is a homeomorphism onto its image. Moreover, $J_{e^2R}(f) \subset \theta(J_R(g))$ and for every $z \in J_R(g)$, $\theta(z) \in \overline{A(\lambda^2|z|, \lambda^{-2}|z|)}$. In particular, $\theta(I_R(g)) \subset I(f)$.

Proof. Let us fix any constant R as in the statement and let $Q := \log(R)$. Note that by assumption, $Q > 2|\log\lambda| + 8\pi + \log L$, and hence, we can apply Theorem 4.44 to the logarithmic transforms F and G defined in 4.43. In particular, by (4.4.9) and the lower bound on Q, it holds that

$$\Theta(J_Q(G)) \subset \mathbb{H}_{Q-2|\log\lambda|} \subset \mathbb{H}_{\log L+8\pi} \cap J(F).$$

In addition, by the commutative relation in (4.4.9), $F(\Theta(J_Q(G))) \subset \Theta(J_Q(G)) \subset J(F)$. Hence, since F is a logarithmic transform of f, by Observation 2.34 applied to the set $\Theta(J_Q(G))$, it holds that $\exp(\Theta(J_Q(G))) \subset J(f) \cap \mathcal{T}_f$. Moreover, since g is of disjoint type, by (2.5.2), $\exp(J(G)) = J(g)$. Hence, by all of the above and since the map Θ is $2\pi i$-periodic, there exists a map $\theta\colon J_R(g) \to J(f)$ defined by the relation $\exp\circ\Theta = \theta\circ\exp$. Then,

$$\theta\circ g\circ\exp = \theta\circ\exp\circ G = \exp\circ\Theta\circ G = \exp\circ F\circ\Theta = f\circ\exp\circ\Theta = f\circ\theta\circ\exp,$$

and since exp is a continuous surjective map, (4.4.10) holds. This is reflected in the following diagram:

$$\begin{tikzcd}
J_Q(G) \arrow[r, "\exp"] \arrow[d, "\Theta"'] \arrow[rrr, bend left, "G"] & J_R(g) \arrow[r, "g"] \arrow[d, "\theta"'] & J_R(g) \arrow[d, "\theta"] & J_Q(G) \arrow[l, "\exp"'] \arrow[d, "\Theta"] \\
J(F) \arrow[r, "\exp"] \arrow[rrr, bend right, "F"'] & J(f) \arrow[r, "f"] & J(f) & J(F). \arrow[l, "\exp"']
\end{tikzcd}$$

By (4.4.9), for any $w \in J_Q(G)$,

$$\{w, \Theta(w)\} \subset \overline{V(\operatorname{Re}(w) - 2|\log\lambda|, \operatorname{Re}(w) + 2|\log\lambda|)}.$$

Hence, if $z \in J_R(g)$, then both $z, \theta(z) \in \overline{A(\lambda^2|z|, \lambda^{-2}|z|)}$, and in particular, $\theta(I_R(g)) \subset I(f)$. □

Next, we shall see that the conjugacy in Corollary 4.45 establishes a convenient relation between the respective external addresses of the maps involved.

4.46 (Fixing addresses for f and g)**.** Let $f \in \mathcal{CB}$ and let g be the disjoint type function specified 4.43, with respective sets of tracts $\mathcal{T}_f$ and $\mathcal{T}_g$. Let us choose a curve $\delta \subset \mathbb{C} \setminus (\mathcal{T}_f \cup \mathcal{T}_g \cup \mathbb{D}_L)$ joining $\partial\mathbb{D}_L$ to infinity. We define fundamental domains for f and g as the connected components of the respective preimages of $\mathbb{C} \setminus (\overline{D} \cup \delta)$ under f and g, and we define external addresses for f and g as sequences of the just defined fundamental domains, see Definition 2.21. As usual, $\operatorname{Addr}(f)$ and $\operatorname{Addr}(g)$ denote the sets of respective admissible external addresses for f and g.

Proposition 4.47 (θ_R relates $\operatorname{Addr}(g)$ and $\operatorname{Addr}(f)$)**.** *Under the conditions in 4.46, for each constant R sufficiently large, the map $\theta := \theta_R$ from Corollary 4.45 establishes a bijection between $\operatorname{Addr}(f)$ and $\operatorname{Addr}(g)$ that preserves their cyclic orders defined in (2.4.8). Moreover, if $J^g_{\underline{s}}$ is a Julia constituent of $J(g)$ for some $\underline{s} \in \operatorname{Addr}(g)$, then $\theta(J^g_{\underline{s}}) \subset J^f_{\underline{\tau}}$ for some $\underline{\tau} \in \operatorname{Addr}(f)$, where $J^f_{\underline{\tau}}$ is the corresponding Julia constituent of f.*

Remark. By the first claim of the proposition, from now on we will assume that in this setting,

$$\theta(J^g_{\underline{s}}) \subset J^f_{\underline{s}}.$$

Moreover, with some abuse of notation and since it will be clear from the context, we drop the dependence of Julia constituents on their functions. That is, we will write $\theta(J_{\underline{s}}) \subset J_{\underline{s}}$.

Proof of Proposition 4.47. Fix any $R > \exp(2|\log\lambda| + 8\pi + L)$ and let $\operatorname{Addr}(J(F))$ and $\operatorname{Addr}(J(G))$ be the sets of external addresses for F and G defined with respect to the sets $\mathcal{T}_F$ and $\mathcal{T}_G$ of logarithmic tracts. See Definition 2.35. Let Θ be the map from Theorem 4.44 for the constant $Q = \log R$. Then, it is stated in [Rem09, Proof of Theorem 3.2] that by construction, for every $w \in J_R(G)$, the external address $\tilde{\underline{s}}$ of $\Theta(w)$ is uniquely determined by the external address $\underline{s}$ of w. More precisely, if $\underline{s} = T_0T_1,\dots$, then $\tilde{\underline{s}} = \tilde{T}_0\tilde{T}_1,\dots$, where each $\tilde{T}_j \in \mathcal{T}_{\mathcal{F}}$ equals $T_j - |\log\lambda|$. In particular, the map Θ acts as a bijection between $\operatorname{Addr}(J(F))$ and $\operatorname{Addr}(J(G))$. Recall that $\exp(\mathcal{T}_F) = \mathcal{T}_f$ and $\exp(\mathcal{T}_G) = \mathcal{T}_g$, and by Observation 2.36, there exist bijections between $\operatorname{Addr}(J(F))/\!\sim$ and $\operatorname{Addr}(f)$, and $\operatorname{Addr}(J(G))/\!\sim$ and $\operatorname{Addr}(g)$, where $\sim$ is the equivalence relation that identifies the first coordinate of external addresses of logarithmic transforms. See (2.5.4). Since the map Θ preserves the vertical order at infinity of connected components of $J(F)$ and $\exp \circ \Theta = \theta \circ \exp$, the map θ preserves the cyclic order at infinity of Julia constituents, and thus, by (2.4.9), the first claim follows. The second part, that is, the fact that for each $\underline{s} \in \operatorname{Addr}(g)$, $\theta(J^g_{\underline{s}}) \subset J^f_{\underline{\tau}}$ for some $\underline{\tau} \in \operatorname{Addr}(f)$, follows from the choice of the curve δ that determines fundamental domains for f and g in 4.46 so that $\delta \cap (\mathcal{T}_f \cup \mathcal{T}_g) = \emptyset$, together with $\theta(J_R(g)) \subset (\mathbb{C} \setminus \mathbb{D}_{\lambda^2 R}) \cap J(f) \subset (\mathbb{C} \setminus \mathbb{D}_L) \cap J(f)$ and the commutative relation $\theta \circ g = f \circ \theta$. $\square$

Observation 4.48 (Different choice of $\operatorname{Addr}(f)$)**.** We note that if instead of defining external addresses for f using the tracts $\mathcal{T}_f$ and fundamental domains specified in 4.46, tracts and external addresses are defined for f with respect to a domain D such that $S(f) \Subset D \subset \mathbb{D}_L$, then Proposition 4.47 still holds for that new set of addresses. The reason for this is that in that case, if $\mathcal{T}_D := f^{-1}(\mathbb{C} \setminus \overline{D})$ is the new set of tracts, then $\mathcal{T}_f \subset \mathcal{T}_D$ and thus, up to a convenient choice of the curve δ in the definition of fundamental domains, Julia constituents with respect to the tracts $\mathcal{T}_f$ are contained in Julia constituents with respect to the tracts $\mathcal{T}_D$.

We note that Theorem 4.44 and Corollary 4.45 do not require the function f to be in class $\mathcal{CB}$ but only in class $\mathcal{B}$, and thus $J(g)$ being a Julia Cantor bouquet has not been used yet. We will do so now to show that with this further assumption, the semiconjugacy near infinity established for f and g in Corollary 4.45 imply that dynamic rays also exist for f.

Proposition 4.49 (θ_R determines a valid initial configuration)**.** *Let $f \in \mathcal{CB}$ and let g be the disjoint type function defined in 4.43. Suppose that external addresses have been defined for f and g following 4.46. For each $R > \exp(2|\log \lambda| + 8\pi + L)$, let $\theta := \theta_R$ be the map from Corollary 4.45 and let $\pi := \pi_{_R} \colon J(g) \to J_R(g)$ be the map defined in (4.3.3). Then, for each $\underline{s} \in \operatorname{Addr}(g)$, the set*

$$\gamma^0_{\underline{s}} := \theta(\pi(J_{\underline{s}})) \tag{4.4.11}$$

is either a ray tail of f or a dynamic ray together with its endpoint. Moreover, $\{\gamma^0_{\underline{s}}\}_{\underline{s} \in \operatorname{Addr}(f)}$ is a valid initial configuration for f in the sense of Definition 4.6.

Remark. We have defined the curves $\gamma^0_{\underline{s}}$ using the map π just for convenience in future references, namely, we will use them in §4.6 to prove Theorem 1.8. However, we note that the same result holds if instead, for each $\underline{s} \in \operatorname{Addr}(g)$ we define $\gamma^0_{\underline{s}} := \theta(J_{\underline{s}})$.

Proof of Proposition 4.49. Since $f \in \mathcal{CB}$ and g is a disjoint type function on its parameter space, by Corollary 4.37 $J(g)$ is a Cantor bouquet and by Proposition 4.27, each of its Julia constituents $J_{\underline{s}}$ is a dynamic ray together with its endpoint. Let us fix some constant R as in the statement. Then, by Theorem 4.31, for each $\underline{s} \in \operatorname{Addr}(g)$, $\pi(J_{\underline{s}})$ is either a ray tail or a dynamic ray together with its endpoint, the latter case occurring whenever $J_{\underline{s}} \subset J_R(g)$. Using that by Corollary 4.45 $f \circ \theta = \theta \circ g$ in $J_R(g)$ and that $\theta|_{I(g)}$ is injective, we can transfer this property to $\gamma^0_{\underline{s}} := \theta(\pi(J_{\underline{s}}))$. That is, $\gamma^0_{\underline{s}}$ is either a ray tail or dynamic ray with its endpoint.

We shall next see that $\{\gamma^0_{\underline{s}}\}_{\underline{s} \in \operatorname{Addr}(f)}$ is a valid initial configuration for f. First, by Proposition 4.47, $\gamma^0_{\underline{s}} \subset J_{\underline{s}}$ for each $\underline{s} \in \operatorname{Addr}(f)$, and in fact, since $\gamma^0_{\underline{s}}$ is a ray tail or dynamic ray with its endpoint, $\gamma^0_{\underline{s}} \subset J^\infty_{\underline{s}}$, the latter set defined in 2.24. Moreover, by Observation 4.32, $g(\pi(J_{\underline{s}})) \subseteq \pi(J_{\sigma(\underline{s})})$, and this together with Corollary 4.45 implies that

$$f(\gamma^0_{\underline{s}}) = (f \circ \theta \circ \pi)(J_{\underline{s}}) = (\theta \circ g \circ \pi)(J_{\underline{s}}) \subseteq \theta(\pi(J_{\sigma(\underline{s})})) = \gamma^0_{\sigma(\underline{s})}.$$

We are left to show that all points in $I(f)$ are eventually mapped to a curve in $\{\gamma^0_{\underline{s}}\}_{\underline{s}\in \mathrm{Addr}(f)}$. By Observation 4.30, for all constants $Q > R$ sufficiently large, $J_Q(g) \subset \pi(J(g))$. Let us fix any such constant Q. If we show that $\theta(J_Q(g)) \supset J_M(f)$ for some $M > 0$, then the result follows, since we would have that

$$J_M(f) \subset \theta(J_Q(g)) \subset \theta(\pi(J(g))) = \left(\bigcup_{\underline{s}\in \mathrm{Addr}(f)} \gamma^0_{\underline{s}} \right),$$

and by definition, if $z \in I(f)$, then $f^n(z) \in J_M(f)$ for some $n \in \mathbb{N}$. It follows from [Rem09, proof of Theorem 3.2] that the map θ in Corollary 4.45 is defined in the same manner for all $Q > \exp(2|\log \lambda| + 1)$, and hence, it holds that for all $Q > R$, $\theta(J_Q(g)) \supset J_{e^2 Q}(f)$ by applying Corollary 4.45 for the constant Q. Alternatively, this can also be seen directly using that by Corollary 4.45, for all $z \in J_R(g)$,

$$z, \theta(z) \in \overline{A(\lambda^2|z|, \lambda^{-2}|z|)}. \tag{4.4.12}$$

More specifically, let us choose $Q > e^2 R$ and let $z \in J_{\lambda^{-2}Q}(f)$. In particular, since $|\lambda| < 1$, $z \in J_{e^2 R}(f)$, and so there exists $w \in J_R(g)$ such that $\theta(w) = z$. Since for all $n \geq 0$, $|f^n(z)| = |\theta(g^n(w))| > \lambda^{-2}Q$, by (4.4.12) it must occur that $|g^n(w)| > Q$, and so $w \in J_Q(g)$. Thus, $J_{\lambda^{-2}Q}(f) \subset \theta(J_Q(g))$, as we wanted to show. $\square$

As a consequence of the previous Proposition 4.49 and Observation 4.7, we have *criniferousness* for f:

Corollary 4.50 (Maps in $\mathcal{CB}$ are criniferous)**.** *If $f \in \mathcal{CB}$, then f is criniferous.*

Proof of Theorem 1.7. It is a direct consequence of Corollary 4.50 and Proposition 4.41. $\square$

4.5 A model space for functions in $\mathcal{CB}$

So far, gathering together results from previous sections, we have shown that if $f \in \mathcal{CB}$, then

(1) f is criniferous (Corollary 4.50), and hence when $\operatorname{AV}(f) \cap J(f) = \emptyset$, its escaping set $I(f)$ consists of a collection of canonical tails indexed by $\operatorname{Addr}(f)_{\pm}$ (Proposition 4.8 and Observation 4.12).

(2) f is semiconjugate to a disjoint type function g from its parameter space in subsets of their Julia sets whose orbits stay in a neighbourhood of infinity, Corollary 4.45.

Recall that our ultimate goal is to define a model for the action of f in $J(f)$, Theorem 1.8. Ideally, we would extend the semiconjugacy in (2) to the whole Julia set of a disjoint type function, a strategy used previously for other classes of functions, for example in [Rem09, MB12]. However, as reflected in (1), the presence of critical values, and hence critical points, in $I(f)$, provides the escaping set with a topological structure different from a (Pinched) Cantor bouquet. Nonetheless, (1) suggests the use of two copies of $J(g)$, say $J(g) \times \{-,+\}$, as a candidate model space: then, in a very rough sense, a function φ could map $J(g) \times \{-,+\}$ to $J(f)$ by mapping $J(g) \times \{+\}$ to those canonical tails with signed addresses with second coordinate "+", and $J(g) \times \{-\}$ to canonical tails whose signed address has second coordinate "−". We will proceed this way in Section 4.6.

Note that for the function φ from Theorem 1.8 to be continuous, we need to provide the set $J(g) \times \{-,+\}$ with the "right topology". That is, we want to use the map θ from Corollary 4.45 to conjugate near infinity each copy of $J(g) \times \{-,+\}$ to a subset of $J(f)$, and since by Proposition 4.47, the function θ preserves the cyclic orders in $\operatorname{Addr}(g)$ and $\operatorname{Addr}(f)$, see 2.31, we must endow $J(g) \times \{-,+\}$ with a topology that is compatible with that of $\operatorname{Addr}(f)_{\pm}$ defined in 4.2. This is our main task in this section. Even if a topology could be defined directly over $J(g) \times \{-,+\}$, for convenience and simplification of arguments, we instead define it in two copies of any straight brush B, and using the corresponding ambient homeomorphism $\psi\colon J(g) \to B$, we induce a topology in our model set, see 4.54.

For the rest of the section, let us fix any $f \in \mathcal{CB}$ and a disjoint type function g from its parameter space. Let us moreover assume that the topological space $\operatorname{Addr}(g)_{\pm}$ has been defined following Definition 4.3. Recall from Definition 4.25 that a straight brush is defined as a subset of $[0,\infty) \times \mathbb{R} \setminus \mathbb{Q}$. Hence, we consider the set

$$\mathcal{M} := [0,\infty) \times \mathbb{R} \setminus \mathbb{Q} \times \{-,+\}, \tag{4.5.1}$$

that we aim to endow with a topology. We will use the symbols "$*, \star, \circledast$" to refer to generic elements of $\{-,+\}$.

4.51 (Topology in $(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$)**.** We start by providing $(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$ with a topology compatible with that of $\operatorname{Addr}(g)_\pm$. Let $<_i$ be the usual linear order on irrationals, and let us give the set $\{-,+\}$ the order $\{-\}\prec\{+\}$. Then, for elements in the set $(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$, we define the order relation

$$(r,*)<(s,\star) \qquad \text{if and only if} \qquad r<_i s \quad \text{or} \quad r=s \ \text{ and } \ *\prec\star. \tag{4.5.2}$$

This gives a total order to $(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$. Thus, we can define a cyclic order induced by "$<$" in the usual way: for $a,x,b\in(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$,

$$[a,x,b]_I \quad \text{if and only if} \quad a<x<b \quad \text{or} \quad x<b<a \quad \text{or} \quad b<a<x,$$

where the subindex "I" stands for *irrationals* of the model. Moreover, given two different elements $a,b\in(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$, we define the *open interval* from a to b, denoted by (a,b), as the set of all points $x\in(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$ such that $[a,x,b]$. The collection of all such open intervals forms a base for the *cyclic order topology*, that we denote by τ_I.

Before we proceed to define a topology in $\mathcal{M}$, let us check that indeed the topological spaces $(\operatorname{Addr}(g)_\pm,\tau_A)$ and $(\mathbb{R}\setminus\mathbb{Q}),\tau_I)$ are closely related.

Proposition 4.52 (Correspondence between spaces)**.** *Let $\psi\colon J(g)\to B$ be an ambient homeomorphism, and for each $\underline{s}\in\operatorname{Addr}(g)$, let $Irr(\underline{s}):=y$, where y is the irrational so that $\psi(J_{\underline{s}})=[t_y,\infty)\times\{y\}\subset B$. Let $\mathcal{C}:\operatorname{Addr}(g)_\pm\to(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$ given by $\mathcal{C}((\underline{s},*))=(Irr(\underline{s}),*)$. Then $\mathcal{C}$ is an open map.*

Proof. Let $\underline{s},\underline{\tau},\underline{\alpha}\in\operatorname{Addr}(g)$. Let $[\cdot]_\ell$ denote the cyclic order on $\operatorname{Addr}(g)$ defined in 2.31. Then,

$$[\underline{s},\underline{\tau},\underline{\alpha}]_\ell \overset{(1)}{\Longleftrightarrow} [J_{\underline{s}},J_{\underline{\tau}},J_{\underline{\alpha}}]_\infty \overset{(2)}{\Longleftrightarrow} [\psi(J_{\underline{s}}),\psi(J_{\underline{\tau}}),\psi(J_{\underline{\alpha}})]_\infty \overset{(3)}{\Longleftrightarrow} [\operatorname{Irr}(\underline{s}),\operatorname{Irr}(\underline{\tau}),\operatorname{Irr}(\underline{\alpha})]_i$$

where (1) is the last claim in 2.31, (2) is by ψ being a homeomorphism and hence preserving the cyclic order at infinity, and (3) is by defining a cyclic order in the irrationals from the usual linear order. Then, if we respectively cut the cyclic orders $[\cdot]_\ell$ and $[\cdot]_i$ in some external address $\underline{s}$ and $\operatorname{Irr}(\underline{s})$, since the linear orders in $\operatorname{Addr}(g)_\pm$ and $(\mathbb{R}\setminus\mathbb{Q})\times\{-,+\}$ are respectively defined in (4.1.2) and (4.5.2) the same way, it follows that

$$[(\underline{s},*),(\underline{\alpha},\star),(\underline{\tau},\circledast)]_A \quad \text{if and only if} \quad [\mathcal{C}((\underline{s},*)),\mathcal{C}((\underline{\alpha},\star)),\mathcal{C}((\underline{\tau},\circledast))]_I. \tag{4.5.3}$$

Then, since we have used these orders to define the respective cyclic order topologies τ_A and τ_I in the respective domain and codomain of $\mathcal{C}$, it is an open map. $\square$

We observe some properties of the topological space defined in 4.51.

Observation 4.53 (Open and closed sets in $(\mathbb{R} \setminus \mathbb{Q} \times \{-,+\}, \tau_I)$)**.** Let A be an open set of $(\mathbb{R} \setminus \mathbb{Q} \times \{-,+\}, \tau_I)$ and suppose that $(s,-),(s,+) \in A$. Then, since τ_I is generated by open intervals, there exist irrationals $r <_i s <_i t$ such that $((r,*),(s,+)) \ni (s,-)$ and $((s,-),(t,*)) \ni (s,+)$. Hence, both $(s,-),(s,+) \in ((r,*),(t,*)) \subset A$. Moreover, for any pair $r <_i s$, the sets $U := ((r,+),(s,-))$, $U \cup \{(r,+)\}$, $U \cup \{(s,-)\}$ and $U \cup \{(r,+),(s,-)\} = ((r,-),(s,+)) =: V$ are open intervals. In addition, V is also closed, since it contains its boundary points.

Remark (Not second countable). The space $(\mathbb{R} \setminus \mathbb{Q} \times \{-,+\}, \tau_I)$ is not second-countable. That is, it is not possible to find a countable collection $\mathcal{U}$ of open sets of τ_I such that any open set in τ_I can be written as a union of some elements in $\mathcal{U}$. To see this, suppose such base $\mathcal{U}$ existed. Since τ_I is generated by intervals, we may assume without loss of generality that all elements in $\mathcal{U}$ are intervals. In particular, they are determined by their pair of boundary points. Let us consider for each $s \in \mathbb{R} \setminus \mathbb{Q}$ the disjoint intervals $I_s^+ := ((s,-),(s+1,+)) \ni (s,+)$ and $I_s^- := ((s-1,+),(s,+)) \ni (s,-)$. Then, there must exist two disjoint elements $U_s^+, U_s^- \in \mathcal{U}$ such that $U_s^+ \subset I_s^+$ and $U_s^- \subset I_s^-$ and such that one endpoint of U_s^- is $(s,-)$ and one endpoint of U_s^+ is $(s,+)$. Then, $\{I_s\}_{s \in \mathbb{R}\setminus\mathbb{Q}}$ is an uncountable collection of intervals, each of them determining two elements of $\mathcal{U}$ with some specific endpoint, different for each $I_{\underline{s}}$ and contradicting that $\mathcal{U}$ is countable.

4.54 (Definition of topologies)**.** Let $\mathcal{M}$ be the set from (4.5.1). We define the topological space $(\mathcal{M}, \tau_{\mathcal{M}})$ with $\tau_{\mathcal{M}}$ being the product topology of $[0,\infty)$ with the usual topology, and $(\mathbb{R} \setminus \mathbb{Q}) \times \{-,+\}$ with the topology τ_I. Let B and ψ be a straight brush and usual ambient homeomorphism for which $\psi(J(g)) = B$. Let $B_\pm := B \times \{-,+\}$ be the subspace of $\mathcal{M}$ with the induced topology $\tau_{B_\pm}$ from $\tau_{\mathcal{M}}$. Consider the set $J(g)_\pm := J(g) \times \{-,+\}$ and the bijection $\tilde{\psi} : J(g)_\pm \to B_\pm$ defined as $\tilde{\psi}((z,*)) := (\psi(z),*)$. We can then induce a topology in $J(g)_\pm$ from the space $(B_\pm, \tau_{B_\pm})$, namely

$$\tau_J := \{\tilde{\psi}^{-1}(U) : U \in \tau_{B_\pm}\}. \tag{4.5.4}$$

Note that in particular, $\tilde{\psi} : (J(g)_\pm, \tau_J) \to (B_\pm, \tau_{B_\pm})$ is a homeomorphism. We moreover define $I(g)_\pm := I(g) \times \{-,+\} \subset J(g)_\pm$ as a subspace equipped with the induced topology.

Definition 4.55 (Model for functions in $\mathcal{CB}$)**.** Let $f \in \mathcal{CB}$ and let g be any disjoint type function on its parameter space. Then, the space $(J(g)_\pm, \tau_J)$ with τ_J defined following 4.54 is a *model space* for f. Moreover, we define its *associated model function* $\tilde{g} : (J(g)_\pm, \tau_J) \to (J(g)_\pm, \tau_J)$ as $\tilde{g}(z, *) := (g(z), *)$.

Observation 4.56 (All models for a fixed function are conjugate)**.** Let $f \in \mathcal{CB}$ and let g_1 and g_2 be two disjoint type functions on its parameter space. Let $J(g_1)_\pm$, $J(g_2)_\pm$ and $\tilde{g}_1$, $\tilde{g}_2$ be the corresponding models and respective associated model functions. Then, there exists a homeomorphism $\Phi : J(g_1)_\pm \to J(g_2)_\pm$ such that $\Phi \circ \tilde{g}_1 = \tilde{g}_2 \circ \Phi$. To see this, note that since any two straight brushes are ambiently homeomorphic, we may assume w.l.o.g that the topologies in $J(g_1)_\pm$ and $J(g_2)_\pm$ have been induced from the same space $B_\pm$ following 4.54. By Corollary 4.37, there exists a homeomorphism $\varphi : J(g_1) \to J(g_2)$ such that $\varphi \circ g_1 = g_2 \circ \varphi$. Defining $\Phi(z, *) := (\varphi(z), *)$ our claim follows.

Remark. Since by the remark preceding 4.54, unlike $\mathbb{C}$, $(\mathcal{M}, \tau_{\mathcal{M}})$ is not a second countable space, it cannot be (topologically) embedded on the plane. By a similar argument, because of the third property in the definition of straight brush, nor can $(B_\pm, \tau_B)$, and consequently nor $(J(g)_\pm, \tau_J)$. Nonetheless, consider any open set U of $\mathcal{M}$ of the form $U := (t_1, t_2) \times (x, y)$, with $t_1 < t_2$ and $x = (r, *)$, $y = (s, \star) \in (\mathbb{R} \setminus \mathbb{Q}) \times \{-, +\}$ for some $r <_i s$. Then, the interval (x, y) comprises all elements $(\alpha, *)$ with $r <_i \alpha <_i s$, and hence we can think of U as being a sort of "box". This intuition will become clearer in the proof of the next proposition.

Proposition 4.57 (Continuity of functions from the model space)**.** *Let $f \in \mathcal{CB}$, and let $J(g)_\pm$ be a model space for f. Then, both its associated model function $\tilde{g}$ and the function* $\operatorname{Proj}\colon J(g)_\pm \to J(g)$ *given by* $\operatorname{Proj}(z, *) := z$ *are continuous.*

Proof. Let $\tilde{\psi} : J(g)_\pm \to B_\pm$ and $\psi\colon J(g) \to B$ be the homeomorphisms from 4.54. Then, proving continuity of Proj is equivalent to proving continuity of the map $\mathcal{P} := (\psi \circ \operatorname{Proj} \circ \tilde{\psi}^{-1})\colon B_\pm \to B$, and hence we do the latter. For any $(t, r, *) \in B_\pm$,

$$\mathcal{P}(t, r, *) = (\psi \circ \operatorname{Proj} \circ \tilde{\psi}^{-1})(x) = (\psi \circ \operatorname{Proj})(\psi^{-1}(x), *) = (\psi \circ \psi^{-1})(x) = (t, r). \tag{4.5.5}$$

Fix $x = (t, r, *) \in B_\pm$, any $\epsilon > 0$ and let $\mathbb{D}_\epsilon(t, r)$ be the (Euclidean) ball of radius ϵ centred at $\mathcal{P}(x)$. We can find a pair of irrational numbers $r_1 < r < r_2$ such that the rectangle $(t - \epsilon/2, t + \epsilon/2) \times (r_1, r_2) \subset \mathbb{D}_\epsilon(t, r)$. Then, $R := ((t - \epsilon/2, t + \epsilon/2) \times ((r_1, +), (r_2, -)) \cap B_\pm)$ is an open subset of $B_\pm$ containing x and such that

$$\mathcal{P}\,(R) = (t - \epsilon/2, t + \epsilon/2) \times (r_1, r_2) \subset \mathbb{D}_\epsilon(t, r),$$

and so $\mathcal{P}$ is continuous. Similarly, proving that $\tilde{g} : J(g)_\pm \to J(g)_\pm$ is continuous is equivalent to proving that $\tilde{h} := \tilde{\psi} \circ \tilde{g} \circ \tilde{\psi}^{-1} : B_\pm \to B_\pm$ is continuous. For any $x = (t, r, *) \in B_\pm$,

$$\tilde{h}(x) = (\tilde{\psi} \circ \tilde{g} \circ \tilde{\psi}^{-1})(x) = (\tilde{\psi} \circ \tilde{g})(\psi^{-1}(t, r), *) = ((\psi \circ g \circ \psi^{-1})(t, r), *).$$

That is, $\tilde{h}(t, r, *) = (h(t, r), *)$, where $h := \psi \circ g \circ \psi^{-1} : B \to B$ is a continuous function.

Fix $x \in B_\pm$ and let V_x be an open neighbourhood of $\tilde{h}(x) =: (t, \alpha, *)$. We may assume without loss of generality that V_x is of the form $V_x := (t_1, t_2) \times (w, y)$, with $t_1 < t < t_2 \in \mathbb{R}$ and $w = (r, \circledast), y = (s, \star) \in B_\pm$ such that $r \lesssim_i \alpha \lesssim_i s$. Let $\mathcal{P} \colon B_\pm \to B$ be the function specified in (4.5.5). If, $r \lnsim_i \alpha \lnsim_i s$, then $(t, \alpha, -), (t, \alpha, +) \in V_x$ and by Observation 4.53, there exists a pair of irrationals α^-, α^+ so that $r \leq \alpha^- < \alpha < \alpha^+ \leq s$ and $H := (t_1, t_2) \times ((\alpha^-, +), (\alpha^+, -)) \subset V_x$. In particular, $\mathcal{P}(H)$ is open and $(\mathcal{P}^{-1} \circ \mathcal{P})(H) = H$. Since both h and $\mathcal{P}$ are continuous functions, $(\mathcal{P}^{-1} \circ h^{-1} \circ \mathcal{P})(H)$ is an open set in $B_\pm$, and by construction $(\tilde{h} \circ \mathcal{P}^{-1} \circ h^{-1} \circ \mathcal{P})(H) \subset V_x$. Otherwise, either $r = \alpha$, which implies that for V_x being an open neighbourhood of $\tilde{h}(x)$, w must be of the form $w = (r, -)$ and $\tilde{h}(x) = (t, r, +)$, or by the same reasoning, $y = (s, +)$ and $\tilde{h}(x) = (t, s, -)$. We only argue continuity in the first case and remark that the second case can be dealt with analogously. Define $R := (t_1, t_2) \times (r, -)$ and $H := (t_1, t_2) \times ((r, +), (s, \star)) \subset V_x$. Note that $\mathcal{P}(H)$ is an open set and $\mathcal{P}(R) \subset \partial\mathcal{P}(H)$. Since g is of disjoint type, $J(g) \cap \operatorname{Crit}(g) = \emptyset$, and so g is locally injective in $J(g)$. Therefore, so is h, which implies that h preserves locally the order of the hairs of the straight brush B. Consequently, $(h^{-1} \circ \mathcal{P})(R) \subset \partial(h^{-1} \circ \mathcal{P})(H)$. By construction and a similar argument to that in the previous case, the set $(h^{-1} \circ \mathcal{P})(R) \times \{-\} \cup (\mathcal{P}^{-1} \circ h^{-1} \circ \mathcal{P})(H)$ is an open neighbourhood of x whose image under $\tilde{h}$ lies in V_x, and continuity follows. □

We conclude this section showing some topological properties of model spaces that will be of use to us in Section 4.6 when proving surjectivity of the function φ from Theorem 1.8.

Lemma 4.58 (Compactification of the model space)**.** *Let $f \in \mathcal{CB}$. Then each model space $(J(g)_\pm, \tau_J)$ for f admits the one point (or Alexandroff)-compactification τ_∞. The new compact space $(J(g)_\pm \cup \{\tilde{\infty}\}, \tau_\infty)$ is a sequential space. Moreover, given a sequence $\{x_k\}_{k \in \mathbb{N}} \subset J(g)_\pm \cup \{\tilde{\infty}\}$,*

$$\lim_{k \to \infty} x_k = \tilde{\infty} \quad \iff \quad \lim_{k \to \infty} \operatorname{Proj}(x_k) = \infty. \tag{4.5.6}$$

Proof. We show that $J(g)_\pm$ admits a one-point compactification by proving that $J(g)_\pm$ is a locally compact, Hausdorff space. Equivalently, since these are topological properties (preserved under homeomorphisms), we instead show that a corresponding double brush $(B_\pm, \tau_{B_\pm})$, see 4.54, is locally compact and Hausdorff. Note that the space $(\mathcal{M}, \tau_{\mathcal{M}})$ defined in 4.54 is Hausdorff: for any $(t, s) \in \mathbb{R}^2$,

$$(t, s, -) \in V_- := (t - t/2, t + 1) \times ((s - 1, -), (s, +))$$
$$(t, s, +) \in V_+ := (t - t/2, t + 1) \times ((s, -), (s + 1, +))$$

and $V_- \cap V_+ = \emptyset$. Disjoint neighbourhoods of any pair of points in $\mathcal{M}$ can be constructed similarly. Since being Hausdorff is a hereditary property, $(B_\pm, \tau_{B_\pm})$ is Hausdorff.

We prove local compactness of $(B_\pm, \tau_{B_\pm})$ by showing that for each $x \in B_\pm$ and each open bounded neighbourhood $U_x \ni x$, the closure of U_x in $B_\pm$, that we denote by $\overline{U}_x$, is compact. With that purpose, let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $\overline{U}_x$. By definition, $B_\pm \setminus \overline{U}_x$ is an open set, and so

$$\mathcal{U}' := \{U_i\}_{i\in I} \cup \{B_\pm \setminus \overline{U}_x\}$$

is an open cover of $B_\pm$. Hence, for each $(t, s, *) \in B_\pm$, there exists $U_{(t,s,*)} \in \mathcal{U}'$ such that $(t, s, *) \in U_{(t,s,*)}$. For each $(t, s) \in B$ denote

$$V_{(t,s)} := U_{(t,s,-)} \cup U_{(t,s,+)} \quad \text{and} \quad \mathcal{V} := \{V_{(t,s)}\}_{(t,s)\in B}.$$

Let $\mathcal{P}\colon B_\pm \to B$ be the projection function specified in (4.5.5), and observe that $\mathcal{P}(V_{(t,s)})$ might not be open, but since both $\{(t, s, -), (t, s, +)\} \subset V_{(t,s)}$, by Observation 4.53, $\mathcal{P}(V_{(t,s)})$ always contains an open neighbourhood $W_{(t,s)} \ni (t, s)$, that we take to be $\mathcal{P}(V_{(t,s)})$ when this set is open. Then,

$$\mathcal{W} := \{W_{(t,s)}\}_{(t,s)\in B}$$

forms an open cover of $B \subset \mathbb{R}^2$, and in particular of the closure $\overline{\mathcal{P}(\overline{U}_x)}$. Note that $\mathcal{P}(\overline{U}_x)$ is a bounded set, since $\overline{U}_x$ is bounded and we showed in the proof of Proposition 4.57 that $\mathcal{P}$ is continuous. Since the straight brush $B \subset \mathbb{R}^2$ satisfies the Heine-Borel property, there exists a finite subcover $\tilde{\mathcal{W}} = \{W_k\}_{k\in K} \subset \mathcal{W}$ of $\overline{\mathcal{P}(\overline{U}_x)}$. For each $k \in K$, choose $V_k \in \mathcal{V}$ such that $W_k \subseteq \mathcal{P}(V_k)$ and denote

$$\tilde{\mathcal{V}} := \{V_k\}_{k\in K} \quad \text{and} \quad \tilde{\mathcal{U}} := \{U_{(t,s,*)} \in \mathcal{U} : U_{(t,s,*)} \subset V_k \in \tilde{\mathcal{V}}\}.$$

By Definition, the set $\tilde{\mathcal{V}}$ has the same number of elements as $\tilde{\mathcal{W}}$ does, and $\tilde{\mathcal{U}}$ has at most double, and so a finite number. Thus, if we show that $\tilde{\mathcal{U}}$ is an open subcover of $\overline{U}_x$, then we will have shown that $(B_\pm, \tau_{B_\pm})$ is locally compact. Note that $\mathcal{P}^{-1}(\tilde{\mathcal{W}})$ is an open cover of $\mathcal{P}^{-1}(\overline{\mathcal{P}(\overline{U}_x)}) \supset \overline{U}_x$, and hence so it is $\tilde{\mathcal{V}}$. Therefore, for each $(t, s, *) \in \overline{U}_x$, there exists $k \in K$ such that $(t, s, *) \in V_k = U_{(t,s',+)} \cup U_{(t',s',-)}$ for some $(t', s') \in B$. If both $U_{(t',s',-)} = \{B_\pm \setminus \overline{U}_x\} = U_{(t',s',+)}$, then $V_k \cap \overline{U}_x = \emptyset$, which contradicts $(t, s, *) \in V_k$. Hence, $(t, s, *) \in U_{(t',s',*)} \in \tilde{\mathcal{U}}$ for some $* \in \{-, +\}$, and so $\tilde{\mathcal{U}}$ is an open subcover of $\overline{U}_x$.

We have shown that $B_\pm$ admits a (Hausdorff) one-point compactification, that we denote by $B_\pm \cup \{\tilde{\infty}\}$. We will see that $B_\pm \cup \{\tilde{\infty}\}$ is a sequential space by proving that more generally, it is a first-countable space, i.e., each point of $B_\pm \cup \{\tilde{\infty}\}$ has a countable neighbourhood basis. By definition, the open sets in $B_\pm \cup \{\tilde{\infty}\}$ are all sets that are open in $B_\pm$ together with all sets of the form $(B_\pm \setminus C) \cup \{\tilde{\infty}\}$, where C is any closed and compact set in $B_\pm$. For each $(t, s, *) \in B_\pm$, a local basis can be chosen to be the collection of sets $\{U_n\}_{n\in\mathbb{N}}$ given by

$$U_n := ((t - 1/n, t + 1/n) \times ((s - 1/n, -), (s + 1/n, +))) \cap B_\pm.$$

In order to find a local basis for $\tilde{\infty}$, for each $N \in \mathbb{N}$ let

$$C_N := [0, N] \times ((-N, -), (N, +)),$$

and note that by Observation 4.53, C_N equals its closure. Thus, reasoning as in the first part of the proof of this lemma, one can see that C_N is compact, and therefore $\{(B_\pm \setminus C_N) \cup \{\tilde{\infty}\}\}_{N\in\mathbb{N}}$ forms a local basis for $\{\tilde{\infty}\}$. Thus, we have shown that $B_\pm \cup \{\tilde{\infty}\}$ is a sequential space.

Finally, if $\{x_k\}_{k\in\mathbb{N}} \subset B \cup \{\tilde{\infty}\}$ is a sequence such that $x_k \to \tilde{\infty}$ as $k \to \infty$, then for every $N \in \mathbb{N}$ there exists $K(N) \in \mathbb{N}$ such that for all $k \geq K(N)$, $x_k \in (B_\pm \setminus C_N)$. Hence, for all $k \geq K(N)$, $\mathcal{P}(x_k) \in \mathcal{P}(B_\pm \setminus C_N) \subset \mathbb{R}^2 \setminus ([0, N] \times [-N, N])$. Therefore,

$$N \to \infty \iff K(N) \to \infty \iff x_k \to \tilde{\infty} \iff \mathcal{P}(x_k) \to \infty,$$

and the last claim of the statement follows. $\square$

4.6 The semiconjugacy

We have developed in §4.1-4.5 all the necessary tools to prove, together with the results on orbifold metrics for postcritically separated maps from Chapter 3, a more precise version of Theorem 1.8 in this one, namely Theorem 4.64. We start by bringing together the parameters and functions that will be involved in its proof.

4.59 (Combination of previous results)**.** Let $f \in \mathcal{CB}$ be an arbitrary but fixed, strongly postcritically separated function. Let us fix a pair of orbifolds $\mathcal{O} = (S, \nu)$ and $\widetilde{\mathcal{O}} = (\widetilde{S}, \tilde{\nu})$ dynamically associated to f provided by Proposition 3.10. In particular, by Proposition 3.10(a), $S = \mathbb{C} \setminus \overline{U}$, where $\overline{U}$ is a, possibly empty, compact set. By Observation 3.2 and since $f \in \mathcal{B}$, we can fix $K > 0$ sufficiently large so that

$$\{P_J \setminus I(f), \overline{U}, S(f), 0, f(0)\} \subset \mathbb{D}_K. \tag{4.6.1}$$

Moreover, $f \in \mathcal{CB}$ implies that f is criniferous, see Corollary 4.50, and since f is strongly postcritically separated, by definition, $AV(f) \cap J(f) = \emptyset$. Then, as by Observation 3.2 condition (PF) in §4.2 holds for f, all standing hypotheses for the results in §4.2 are satisfied by f, and thus all the results in that section apply to it. In order to make use of them, we define fundamental hands for f (see Definition 4.17) and the set of external addresses $\operatorname{Addr}(f)$ for f according to 4.19. In particular, tracts for f are defined as preimages of $\mathbb{C} \setminus D$, where the domain D is provided by Proposition 4.16. We denote this set of tracts by $\mathcal{T}_f$. Let us choose a constant $L > K$ such that $D \cup f(\mathbb{D}_K) \Subset \mathbb{D}_L$. Then, since $f \in \mathcal{CB}$, using this constant L in 4.43, we get a disjoint type function

$$g := g_\lambda \quad \text{given by} \quad g_\lambda(z) := f(\lambda z)$$

for the constant $\lambda \in \mathbb{C}^*$ provided by (4.4.6). Hence, Corollary 4.45 applies to f and g, and in particular, we can fix some constant $R > L$ big enough for which Corollary 4.45 provides us with a continuous map

$$\theta := \theta_R \colon J_R(g) \to J(f) \cap \mathcal{T}_f.$$

Moreover, if we define the set $\operatorname{Addr}(g)$ of external addresses for g as specified in 4.46, by Proposition 4.47 and Observation 4.48, the map θ establishes an order-preserving 1-to-1 correspondence between $\operatorname{Addr}(g)$ and $\operatorname{Addr}(f)$. More specifically, for each $\underline{s} \in \operatorname{Addr}(g)$, we can assume that the Julia constituent $J_{\underline{s}}$ of g is mapped under θ to the Julia constituent $J_{\underline{s}}$ of f. See Proposition 4.47. Then, for the fixed constant R we consider the map

$$\pi := \pi_R \colon J(g) \to J_R(g)$$

defined in (4.3.3). By Proposition 4.49, for each $\underline{s} \in \operatorname{Addr}(g)$, the curve

$$\gamma^0_{\underline{s}} := \theta(\pi_R(J_{\underline{s}}))$$

is either a ray tail of f or a dynamic ray together with its endpoint, and the set

$$\left\{\gamma^0_{\underline{s}}\right\}_{\underline{s}\in\operatorname{Addr}(f)} \tag{4.6.2}$$

is a valid initial configuration for f in the sense of Definition 4.6. Let $\operatorname{Addr}(f)_\pm$ be the space of signed external addresses for f defined from $\operatorname{Addr}(f)$, see Definition 4.3. Using the initial configuration in (4.6.2), we define the set of canonical tails

$$\mathcal{C} := \left\{\gamma^n_{(\underline{s},*)} : n \geq 0 \text{ and } (\underline{s},*) \in \operatorname{Addr}(f)_\pm\right\}$$

provided by Proposition 4.8. In particular, by Proposition 4.22 and Lemma 4.23, for each curve $\gamma^n_{(\underline{s},*)} \in \mathcal{C}$, there exists a neighbourhood $\tau_n(\underline{s},*) \supset \gamma^n_{(\underline{s},*)}$ where we can define an inverse branch of f

$$f^{-1,[n]}_{(\underline{s},*)} := \left(f|_{\tau_n(\underline{s},*)}\right)^{-1} : f(\tau_n(\underline{s},*)) \to \tau_n(\underline{s},*), \tag{4.6.3}$$

as well as an inverse branch of f^n provided by Observation 4.24,

$$f^{-n}_{(\underline{s},*)} := \left(f^n|_{\tau_n(\underline{s},*)}\right)^{-1} : f^n(\tau_n(\underline{s},*)) \to \tau_n(\underline{s},*), \tag{4.6.4}$$

having both of them properties we shall use later on. Next, using the function g, we fix a model space $(J(g)_\pm, \tau_J)$ for f (see Definition 4.55) and the corresponding associated model function

$$\tilde{g} \colon J(g)_\pm \to J(g)_\pm \quad \text{ given by } \quad \tilde{g}((z,*)) := (g(z),*).$$

In addition,

$$\operatorname{Proj}\colon J(g)_\pm \to J(g), \quad \text{ with } \quad \operatorname{Proj}(z,*) := z,$$

is the projection function from Proposition 4.57. Finally, we define the set of external addresses $\operatorname{Addr}(g)_\pm$ for g from the space of addresses $\operatorname{Addr}(g)$.

For clarity of exposition, we foliate the model space $J(g)_\pm$ into the sets whose images under the map Proj share the same signed external address. That is, since

g is a disjoint type function, each $z \in J(g)$ belongs to a unique Julia constituent $J_{\underline{s}}$ for some $\underline{s} \in \text{Addr}(g)$, see (2.4.4). This allows us to define analogous sets for all points in $J(g)_\pm$. More precisely:

4.60 (Addressed components of the model)**.** For each $(\underline{s}, *) \in \text{Addr}(g)_\pm$, denote

$$J_{(\underline{s},*)} := \{(z,*) \in J(g)_\pm : z \in J_{\underline{s}}\} \quad \text{and} \quad I_{(\underline{s},*)} := J_{(\underline{s},*)} \cap I(g)_\pm.$$

Moreover, for each $x \in J(g)_\pm$, $\text{addr}(x)$ denotes the unique $(\underline{s}, *) \in \text{Addr}(g)_\pm$ such that $x \in J_{(\underline{s},*)}$.

Remark. Similar notation has been used for two different concepts: in Lemma 4.23, $\mathcal{I}(\underline{s}, *)$ denotes an interval of addresses in $(\text{Addr}(f)_\pm, \tau_A)$, while the set $I_{(\underline{s},*)}$ just defined is a collection of points of the topological space $J(g)_\pm$. Even if this could potentially generate confusion, we have opted to keep this notation, that seems natural, because we believe that their meaning will be clear from the context they appear on. Moreover, we warn the these sets should not be confused with those from Definition 2.21, that is, with Julia constituents.

After setting in 4.59 the functions we use in the proof of Theorem 1.8, for ease of understanding, we now comment on the main ideas of this proof. For the functions f and $\tilde{g}$ fixed in 4.59, we aim to obtain the function $\varphi\colon J(g)_\pm \to J(f)$, that semiconjugates them, as a limit of functions $\varphi_n\colon J(g)_\pm \to J(f)$ that are successively *better approximations* of φ. For each $x \in J(g)_\pm$ and each $n \geq 0$, roughly speaking, $\varphi_n(x)$ is defined the following way: we iterate x under the model function $\tilde{g}$ a number n of times. In particular, if $\text{addr}(x) = (\underline{s}, *)$, then $\tilde{g}^n(x) \subset J_{(\sigma^n(\underline{s}),*)}$. Then, we use the functions Proj, π and θ to *move* from the space $J(g)_\pm$ to the dynamical plane of f. More precisely, if $\text{Proj}(x) = z$, then $\theta(\pi(g^n(z))) \in \gamma^0_{(\sigma^n(\underline{s}),*)}$. Then, we use the composition of n inverse branches of f of the form specified in (4.6.3) to obtain, thanks to Lemma 4.23 and Observation 4.24, a point in $\gamma^n_{(\underline{s},*)}$, that is $\varphi_n(x)$. See Figure 12.

Since we have shown in different propositions throughout the document that all the functions involved in the definition of φ_n are continuous, continuity of φ_n will follow easily in Proposition 4.62. Moreover, we use Theorem 1.1, that is, orbifold expansion of f in a neighbourhood of $J(f)$, to show that the functions φ_n converge to a limit function φ in Lemma 4.63. Finally, since $J(g)$ is a Cantor bouquet and g is of disjoint type, all but some of the endpoints of the hairs of $J(g)$ are escaping (Proposition 4.27). This implies that for each $z \in I(g)$, there exists $N \in \mathbb{N}$ such that $\pi(g^n(z)) = g^n(z)$ for all $n \geq N$, which in turn will allow us to show surjectivity of φ in the proof of Theorem 4.64.

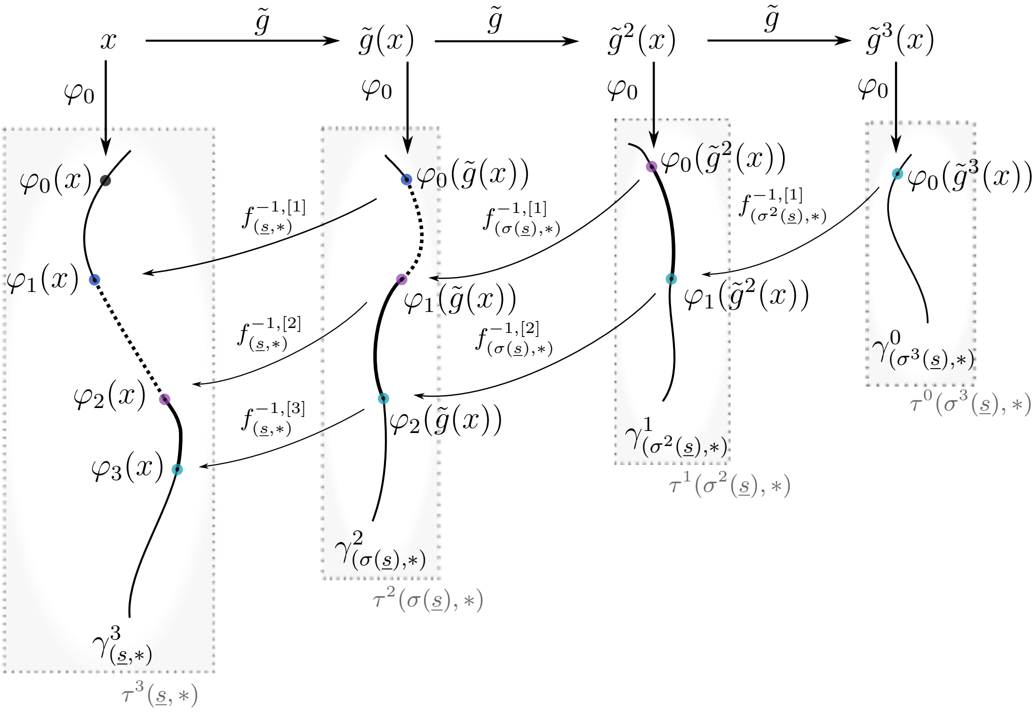

Figure 12: A schematic of the functions and curves involved in the definition of the functions $\{\varphi_n\}_{n\in\mathbb{N}}$.

We now formally define the functions φ_n inductively.

Definition 4.61 (Functions φ_n)**.** Under the assumptions of 4.59, for each $n \geq 0$ we define the function $\varphi_n \colon J(g)_\pm \to J(f)$ as

$$\varphi_0(x) := \theta(\pi(\operatorname{Proj}(x))) \qquad\qquad \varphi_{n+1}(x) := f^{-1,[n]}_{\operatorname{addr}(x)}(\varphi_n(\tilde{g}(x))).$$

We shall now see that these functions are well-defined. The function φ_0 is well-defined since by definition of π, $\pi(\operatorname{Proj}(J(g)_\pm)) \subset J_R(g)$. See Observation 4.30. For each $n \geq 1$, choose any $x \in J(g)_\pm$ and suppose that $\operatorname{addr}(x) = (\underline{s}, *)$. Then, expanding definitions and using Observation 4.24,

$$\begin{aligned}\varphi_n(x) &= \left(f^{-1,[n]}_{\operatorname{addr}(x)} \circ f^{-1,[n-1]}_{\operatorname{addr}(\tilde{g}(x))} \circ \cdots \circ f^{-1,[1]}_{\operatorname{addr}(\tilde{g}^{n-1}(x))} \circ \varphi_0 \circ \tilde{g}^n\right)(x) \\ &= \left(f^{-1,[n]}_{(\underline{s},*)} \circ f^{-1,[n-1]}_{(\sigma(\underline{s}),*)} \circ \cdots \circ f^{-1,[1]}_{(\sigma^{n-1}(\underline{s}),*)} \circ \varphi_0 \circ \tilde{g}^n\right)(x) \\ &= f^{-n}_{(\underline{s},*)}(\varphi_0(\tilde{g}^n(x))).\end{aligned} \tag{4.6.5}$$

Since the equalities in (4.6.5) only depend on $\operatorname{addr}(x)$ but not on the point x itself, the action of φ_n can be expressed in terms of the sets from 4.60 as

$$\varphi_n|_{J_{(\underline{s},*)}} \equiv f^{-n}_{(\underline{s},*)} \circ \varphi_0 \circ \tilde{g}^n|_{J_{(\underline{s},*)}}. \tag{4.6.6}$$

Thus, since each $x \in J(g)_\pm$ belongs to the unique set $J_{(\underline{s},*)}$ for $(\underline{s},*) = \operatorname{addr}(x)$, φ_n is a well-defined function for all $n \geq 0$. In particular, by Observation 4.24, for each $(\underline{s},*) \in \operatorname{Addr}(g)_\pm$,

$$\varphi_n(J_{(\underline{s},*)}) = \gamma^n_{(\underline{s},*)}. \tag{4.6.7}$$

Moreover, by construction, for all $n \geq 0$

$$\varphi_n \circ \tilde{g} = f \circ \varphi_{n+1}. \tag{4.6.8}$$

Proposition 4.62 (Continuity of the functions φ_n)**.** *For each $n \geq 0$, the function $\varphi_n : (J(g)_\pm, \tau_J) \to J(f)$ is continuous.*

Proof. The function φ_0 is continuous because it is the composition of three continuous functions, see Corollary 4.45, Theorem 4.31 and Proposition 4.57. Fix any $n \geq 1$, fix an arbitrary $x \in J(g)_\pm$, let $\operatorname{addr}(x) =: (\underline{s},*)$ and let $\mathcal{I}^n(\underline{s},*)$ be the interval in $\operatorname{Addr}(f)_\pm$ provided by Lemma 4.23. As we noted in 4.59, θ establishes a one-to-one and order-preserving correspondence between $\operatorname{Addr}(f)$ and $\operatorname{Addr}(g)$. Hence, up to this correspondence, the topological spaces $\operatorname{Addr}(f)_\pm$ and $\operatorname{Addr}(g)_\pm$ are the same, and so, $\mathcal{I}^n(\underline{s},*)$ is an open interval in $(\operatorname{Addr}(g)_\pm, \tau_A)$. Let us consider the subset of $J(g)_\pm$

$$A := \bigcup_{(\underline{\tau},\star) \in \mathcal{I}^n(\underline{s},*)} J_{(\underline{\tau},\star)}.$$

Then, by Observation 4.24 and (4.6.6),

$$\varphi_n|_A \equiv f^{-n}_{(\underline{s},*)} \circ \varphi_0 \circ \tilde{g}^n|_A. \tag{4.6.9}$$

It follows that $\varphi_n|_A$ is continuous as it is a composition of continuous functions: we have just shown that φ_0 is continuous, and $\tilde{g}$ is continuous by Proposition 4.57. Moreover, by Observation 4.24 it holds that $\varphi_0(\tilde{g}^n(A)) \subset f^n(\tau_n(\underline{s},*))$, and thus, the restriction of $f^{-n}_{(\underline{s},*)}$ to $\varphi_0(\tilde{g}^n(A))$ is well-defined and continuous.

We are only left to show that A contains an open neighbourhood of x. Recall that we defined in Proposition 4.52 an open map $\mathcal{C} : (\operatorname{Addr}(g)_\pm, \tau_A) \to (\mathbb{R} \setminus \mathbb{Q} \times \{-,+\}, \tau_I)$. Since $\tilde{\psi}(x) = (t, \mathcal{C}(\underline{s},*)) \in B$ for some $t > 0$ and $\mathcal{C}$ is an open map, $\mathcal{C}(\mathcal{I}^n(\underline{s},*))$ is an open interval in $(\mathbb{R} \setminus \mathbb{Q} \times \{-,+\}, \tau_I)$. Then, $U := ((t_1,t_2) \times \mathcal{C}(\mathcal{I}^n(\underline{s},*))) \cap B$ is an open neighbourhood of $\tilde{\psi}(x)$ in B for any choice of $t_1, t_2 \in \mathbb{R}^+$ such that $t_1 < t < t_2$. Consequently, see 4.54, $\tilde{\psi}^{-1}(U)$ is an open neighbourhood of x that lies in A. □

Convergence to the semiconjugacy

By Proposition 3.10, the orbifold $\mathcal{O} \Supset J(f)$ fixed in 4.59 is hyperbolic, and in particular admits an orbifold metric. We denote by $d_{\mathcal{O}}$ the corresponding distance function. Then, we shall now see using results from Chapter 3 that for any given point $x \in J(g)_{\pm}$, $d_{\mathcal{O}}(\varphi_{n+1}(x), \varphi_n(x)) \to 0$ as $n \to \infty$.

Lemma 4.63 (The functions φ_n form a Cauchy sequence)**.** *There exist constants $\mu > 0$ and $\Lambda > 1$ such that for each $x \in J(g)_{\pm}$,*

(A) $d_{\mathcal{O}}(\mathrm{Proj}(x), \varphi_0(x)) < \mu$*,*

(B) $d_{\mathcal{O}}(\varphi_{n+1}(x), \varphi_n(x)) \leq \frac{\mu}{\Lambda^n}$ *for every* $n \geq 0$*.*

Proof. Fix $x \in J(g)_{\pm}$ and let $z := \mathrm{Proj}(x)$. In particular, $\varphi_0(x) = \theta(\pi(z))$. By our choice of the function g in 4.59 and by 4.43, there exists a logarithmic transform G of g satisfying (4.4.8), and in particular, the map g satisfies the assumptions on Theorem 4.31. Consequently, there exists a constant $M > 0$, that does not depend on the point z, such that $\pi(z) \in \overline{A(M^{-1}|z|, M|z|)}$. Moreover, see (4.6.1) and (4.4.7), $J(g) \subset \mathbb{C} \setminus \mathbb{D}_K \subset \mathcal{O}$. Since $\pi(J(g)) \subset J_R(g)$ and by our assumption on the constant R in 4.59, it must hold $R > \lambda^{-2}L$, by Corollary 4.45

$$\theta(\pi(J(g))) \subset \mathbb{C} \setminus \mathbb{D}_{\lambda^2 R} \subset \mathbb{C} \setminus \mathbb{D}_K. \tag{4.6.10}$$

Consequently, $\theta(\pi(J(g))) \cup J(g) \subset \mathbb{C} \setminus \mathbb{D}_K \subset \mathcal{O}$. Moreover, by Corollary 4.45, $\theta(\pi(z)) \in \overline{A(\lambda^2|\pi(z)|, \lambda^{-2}|\pi(z)|)}$, and so

$$\{z,\ \theta(\pi(z))\} \subset A(\lambda^2 M^{-1}|z|, \lambda^{-2}M|z|) \cap (\mathbb{C} \setminus \mathbb{D}_K)\,. \tag{4.6.11}$$

Let us choose a constant $\tilde{K} > 0$ such that

$$\lambda^2 M^{-1} > \tilde{K}. \tag{4.6.12}$$

Then, by Lemma 3.12, there exists $\tilde{R} > 0$ so that if $\overline{A(r, \tilde{K}r)} \subset A(r/\tilde{K}, r\tilde{K}^2) \subset \mathcal{O}$, then the $\mathcal{O}$-distance between any two points in $\overline{A(r, \tilde{K}r)}$ is less than $\tilde{R}$. We want to combine this result with (4.6.11) to get an upper bound for $d_{\mathcal{O}}(z, \theta(\pi(z)))$ by expressing the annulus in (4.6.11) as a finite union of annuli of the form $A(r, \tilde{K}r)$ for some $r > 0$. More specifically, let N the smallest number for which $\tilde{K}^N \geq \lambda^{-4}M^2$. That is, $N := \left\lceil \frac{2\log M - 4\log\lambda}{\log \tilde{K}} \right\rceil$, and let $r := \lambda^2 M^{-1}|z|$. Then, by (4.6.11), (4.6.12) and the choice of K in (4.6.1),

$$\{z,\ \theta(\pi(z))\} \subset \bigcup_{i=1}^{N} \overline{A\left(\tilde{K}^{i-1}r, \tilde{K}^i r\right)} \subset \bigcup_{i=1}^{N} A\left(\tilde{K}^{i-2}r, \tilde{K}^{i+1}r\right) \subset \mathcal{O}. \tag{4.6.13}$$

Thus, since the constant N does not depend on the point $z \in J(g)$, we have that for all $x \in J(g)_\pm$, $d_\mathcal{O}(\varphi_0(x), \operatorname{Proj}(x)) \leq N \cdot \tilde{R} =: \mu_1$, and item (A) is proved.

We now prove item (B). Let $\operatorname{addr}(x) = (\underline{s}, *)$ and fix any $n \in \mathbb{N}$. Recall that by (4.6.7), since by Proposition 4.8 $\gamma^n_{(\underline{s},*)} \subseteq \gamma^{n+1}_{(\underline{s},*)}$,

$$\{\varphi_n(x), \varphi_{n+1}(x)\} \subset \gamma^{n+1}_{(\underline{s},*)}.$$

Thus, the $\mathcal{O}$-length of the piece of $\gamma^{n+1}_{(\underline{s},*)}$ that joins $\varphi_n(x)$ and $\varphi_{n+1}(x)$ provides an upper bound for the $\mathcal{O}$-distance between these two points. Let $\delta(n)$ be that curve. Then, using (4.6.6) and (4.6.8), it holds that

$$f^n(\varphi_n(x)) = \varphi_0(\tilde{g}^n(x)), \qquad f^n(\varphi_{n+1}(x)) = \varphi_1(\tilde{g}^n(x)),$$

and $\delta(1) := f^n(\delta(n)) \subset \gamma^1_{(\sigma^n(\underline{s}),*)}$ is a curve with endpoints $\varphi_0(\tilde{g}^n(x))$ and $\varphi_1(\tilde{g}^n(x))$ and such that $f^{-n}_{(\underline{s},*)}(\delta(1)) = \delta(n)$. See Figure 12. Since $f \in \mathcal{B}$ and is strongly postcritically separated, by Corollary 3.13, any upper bound for $\ell_\mathcal{O}(\delta(1))$ is also an upper bound for $\ell_\mathcal{O}(\delta(n))$. In particular, if we find a constant C that bounds the $\mathcal{O}$-length of the subcurve in $\gamma^1_{\operatorname{addr}(y)}$ between $\varphi_0(y)$ and $\varphi_1(y)$ for all $y \in J(g)_\pm$, being C independent of the point y, then the (B) would follow. However, those subcurves are pieces of ray tails, and in principle might not be rectifiable. Therefore and instead, we find curves in their post-0-homotopy class (see Definition 3.23) with bounded orbifold length. More specifically:

Claim. There exists a constant $\mu_2 > 0$ such that for each $x \in J(g)_\pm$ and $n \geq 0$, if $\delta(1)$ is the piece of $\gamma^1_{\operatorname{addr}(x)}$ joining $\varphi_0(x)$ and $\varphi_1(x)$, then there exists $\tilde{\delta}(1) \in [\delta(1)]_0$ with $\ell_\mathcal{O}(\tilde{\delta}(1)) \leq \mu_2$.

Proof of claim. For an arbitrary $x \in J(g)_\pm$, if $\varphi_0(x) = \varphi_1(x)$, the claim holds trivially. Otherwise, note that expanding definitions and using Corollary 4.45 and (4.6.8),

$$f(\varphi_0(x)) = \theta(g(\pi(\operatorname{Proj}(x)))) \quad \text{ and } \quad f(\varphi_1(x)) = \varphi_0(\tilde{g}(x)) = \theta(\pi(\operatorname{Proj}(\tilde{g}(x)))).$$

Thus, since by Corollary 4.45 θ is a homeomorphism to its image and $f|_{\gamma^1_{\operatorname{addr}(x)}}$ is injective,

$$\varphi_0(x) \neq \varphi_1(x) \iff g(\pi(\operatorname{Proj}(x))) \neq \pi(\operatorname{Proj}(\tilde{g}(x))). \tag{4.6.14}$$

By Observation 4.32, the second relation only occurs if the piece of the Julia constituent $J_{\underline{s}} \subset J(g)$ between the preimage of $\pi(\operatorname{Proj}(\tilde{g}(x)))$ in $J_{\underline{s}}$ and $\pi(\operatorname{Proj}(x))$ is totally contained in the bounded set $\overline{S}_R \supset \mathbb{D}_R$ from the definition of π. Hence, we are aiming to find curves post-0-homotopic to the pieces of dynamic rays totally contained in $\theta(\overline{S}_R \cap J_R(g))$ with uniformly bounded length using Corollary 3.27.

By (4.6.10) and Corollary 4.45, $\theta(\overline{S}_R \cap J_R(g)) \subset \overline{S_R \setminus \mathbb{D}_K} \cap \mathcal{T}_f \subset \mathcal{O}$. Note that $(\overline{S_R \setminus \mathbb{D}_K}) \cap P(f) \subset I(f)$, since by the choice of the constant K in (4.6.1), $(P(f) \setminus I(f)) \subset \mathbb{D}_{K/\tilde{K}} \Subset \mathbb{D}_K$. Moreover, by discreteness of P_J, $\overline{S_R \setminus \mathbb{D}_K} \cap P_J$ is a finite set, and by Corollary 4.9, there exists at least one dynamic ray landing at each point of that intersection. By Proposition 2.19(4), only finitely many pieces of the tracts $\mathcal{T}_f$ intersect $\overline{S_R \setminus \mathbb{D}_K}$, say $\{T_1, \dots, T_m\}$. Each of these pieces T_i is simply-connected and its boundary is an analytic curve, and hence locally connected. Thus, we can apply Corollary 3.27 to the closure of each T_i in $\overline{S_R \setminus \mathbb{D}_K}$ to obtain a constant L_i such that for any (connected) piece of ray tail $\xi \subset \overline{T_i} \cap \overline{S_R \setminus \mathbb{D}_K}$, there exists $\delta \in [\xi]_0$ with $\ell_{\mathcal{O}}(\delta) \leq L_i$. Letting $\mu_2 := \max_{1 \leq i \leq m} L_i$ the claim follows. △

In particular, for each $x \in J(g)_\pm$ and $n \geq 0$, if $\delta(1)$ is the piece of $\gamma^1_{(\underline{s},*)}$ joining $\varphi_0(\tilde{g}^n(x))$ and $\varphi_1(\tilde{g}^n(x))$, then there exists $\tilde{\delta}(1) \in [\delta(1)]_0$ with $\ell_{\mathcal{O}}(\tilde{\delta}(1)) \leq \mu_2$. Hence, by Proposition 3.24, if $\delta(n) \subset f^{-n}_{(\underline{s},*)}(\delta(1))$ is the curve joining $\varphi_0(x)$ and $\varphi_1(x)$, then there exists a unique curve $\tilde{\delta}(n) \subseteq f^{-n}_{(\underline{s},*)}(\tilde{\delta}(1))$ satisfying $\tilde{\delta}(n) \in [\delta(n)]_0$. In particular, $\tilde{\delta}(n)$ has endpoints $\varphi_0(x)$ and $\varphi_1(x)$, and moreover, by Corollary 3.13, there exists a constant $\Lambda > 1$, that does not depend on x, such that

$$d_{\mathcal{O}}(\varphi_{n+1}(x), \varphi_n(x)) \leq \ell_{\mathcal{O}}(\tilde{\delta}(n)) \leq \frac{\ell_{\mathcal{O}}(\tilde{\delta}(1))}{\Lambda^n} \leq \frac{\mu_2}{\Lambda^n}.$$

Letting $\mu := \max\{\mu_1, \mu_2\}$ the lemma follows. □

Finally, we state and prove a more detailed version of Theorem 1.8:

Theorem 4.64. *Let $f \in \mathcal{CB}$ be strongly postcritically separated, let $J(g)_\pm$ be a model space for f and let $\tilde{g}$ be its associated model function. Then there exists a continuous surjective function*

$$\varphi\colon J(g)_\pm \to J(f) \qquad \textit{so that} \qquad f \circ \varphi = \varphi \circ \tilde{g}$$

and $\varphi(I(g)_\pm) = I(f)$. In addition, there exists a constant K such that for every $z \in I(f)$, $\#\varphi^{-1}(z) = \#\operatorname{Addr}(z)_\pm < K$. Moreover, for each $(\underline{s}, *) \in \operatorname{Addr}(g)_\pm$, the restriction $\varphi\colon J_{(\underline{s},*)} \to \overline{\Gamma}(\underline{s}, *)$ is a bijection, and so $\overline{\Gamma}(\underline{s}, *)$ is a canonical ray together with its endpoint.

Observation 4.65. We have implicitly stated in Theorem 4.64 that φ establishes a one-to-one correspondence between $\operatorname{Addr}(g)_\pm$ and $\operatorname{Addr}(f)_\pm$, since with some abuse of notation, we have stated that for each $(\underline{s}, *) \in \operatorname{Addr}(g)_\pm$, $J_{(\underline{s},*)} \subset J(g)_\pm$ is mapped to $\overline{\Gamma}(\underline{s}, *) \subset J(f)$ for $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$. Here, $\overline{\Gamma}(\underline{s}, *)$ denotes the closure of $\Gamma(\underline{s}, *)$ in $\mathbb{C}$. In particular, we are claiming that φ is an order-preserving continuous map.

Proof of Theorem 4.64. Since by Observation 4.56 any two models for f are conjugate, we may assume without loss of generality that g is the disjoint type function fixed in 4.59, and consequently so are $J(g)_\pm$ and $\tilde{g}$. Let $\{\varphi_n\}_{n\geq 0}$ be the sequence of functions given by Definition 4.61 following 4.59. By Proposition 4.62 and Lemma 4.63, $\{\varphi_n\}_{n\geq 0}$ is a uniformly Cauchy sequence of continuous functions. Since the orbifold metric in $\mathcal{O}$ is complete, they converge uniformly to a continuous limit function $\varphi : J(g)_\pm \to \mathcal{O}$, which by the functional equation (4.6.8) satisfies

$$\varphi \circ \tilde{g} = f \circ \varphi. \tag{4.6.15}$$

By Lemma 4.63,

$$\begin{aligned} d_{\mathcal{O}}(\varphi(x), \operatorname{Proj}(x)) &\leq d_{\mathcal{O}}(\varphi(x), \varphi_0(x)) + d_{\mathcal{O}}(\varphi_0(x), \operatorname{Proj}(x)) \\ &\leq \sum_{k=0}^{\infty} d_{\mathcal{O}}(\varphi_{k+1}(x), \varphi_k(x)) + \mu \leq 2\sum_{j=0}^{\infty} \frac{\mu}{\Lambda^j} = \frac{2\mu\Lambda}{\Lambda - 1}. \end{aligned}$$

This means for sequences $\{x_n\}_{n\in\mathbb{N}} \subset J(g)_\pm$ that as $n \to \infty$,

$$\varphi(x_n) \to \infty \quad \text{ if and only if } \quad \operatorname{Proj}(x_n) \to \infty. \tag{4.6.16}$$

In particular, this holds when $\{x_n\}_{n\in\mathbb{N}} = \{\tilde{g}^n(x)\}_{n\in\mathbb{N}}$ is the orbit of some $x \in I(g)_\pm$. Using that by (4.6.15) $\varphi(\tilde{g}^n(x)) = f^n(\varphi(x))$, we have that $x \in I(g)_\pm$ if and only if $\varphi(x) \in I(f)$. Equivalently,

$$\varphi(I(g)_\pm) \subseteq I(f) \quad \text{ and } \quad \varphi(J(f)_\pm \setminus I(g)_\pm) \subseteq J(f) \setminus I(f). \tag{4.6.17}$$

Recall that for each $(s, *) \in \operatorname{Addr}(g)_\pm$, $J_{\underline{s}}$ is a Julia constituent of $J(g)$ and the set $I_{(\underline{s},*)}$ from 4.60 can be expressed as $I_{(\underline{s},*)} = (J_{\underline{s}} \cap I(g)) \times \{*\}$. Recall that since g is a disjoint type function whose Julia set is a Cantor bouquet, by Proposition 4.27, each of its Julia constituents $J_{\underline{s}}$ is a dynamic ray together with its endpoint, and hence, contains at most one non-escaping point, namely its endpoint $e_{\underline{s}}$. Thus, $I_{(\underline{s},*)} =: I_{\underline{s}} \times \{*\}$, where $I_{\underline{s}}$ is either a ray tail or dynamic ray, and for each $* \in \{-, +\}$,

$$J_{(\underline{s},*)} \setminus I_{(\underline{s},*)} \subseteq \{(e_{\underline{s}}, *)\}. \tag{4.6.18}$$

Claim. For each $(s,*) \in \operatorname{Addr}(g)_\pm$, $\varphi|_{I_{(\underline{s},*)}} \to \Gamma(\underline{s},*) \cap I(f)$ is a bijection.

Proof of claim. To prove injectivity of $\varphi|_{I_{(\underline{s},*)}}$, since $I_{\underline{s}}$ can be expressed as the union of a nested sequence of ray tails, it suffices to show that for each ray tail $\xi \subset I_{\underline{s}}$, there exists $N := N(\xi) \in \mathbb{N}$ such that for all $n \geq N$,

$$\varphi_n|_{(\xi\times\{*\})} \text{ is injective} \quad \text{and} \quad \varphi|_{(\xi\times\{*\})} \equiv \varphi_n|_{(\xi\times\{*\})}. \tag{4.6.19}$$

Indeed, if ξ is a ray tail, as by definition it escapes uniformly to infinity, there exists N such that $g^n(\xi) \subset \mathbb{C} \setminus S_R$ for all $n \geq N$, where S_R is the bounded set from the definition of the map π in (4.3.3). In particular, by definition, π acts as the identity map when restricted to $g^n(\xi)$ for all $n \geq N$. Hence, by (4.6.6),

$$\varphi_n|_{(\xi\times\{*\})} = f^{-n}_{(\underline{s},*)} \circ \varphi_0 \circ \tilde{g}^n|_{(\xi\times\{*\})} = f^{-n}_{(\underline{s},*)} \circ \theta \circ g^n|_\xi, \tag{4.6.20}$$

and thus $\varphi_n|_{(\xi\times\{*\})}$ is injective as it is a composition of injective functions, see Proposition 4.27, Corollary 4.45, and Lemma 4.23. Since for all $n \geq N$ the map π acts as the identity map when restricted to $g^n(\xi)$, if we choose any $n_1 \geq n_2 \geq N$, then

$$\varphi_{n_2}|_{(\xi\times\{*\})} = f^{-n_1}_{(\underline{s},*)} \circ f^{n_1-n_2} \circ \theta \circ g^{n_2}|_\xi \overset{(\star)}{=} f^{-n_1}_{(\underline{s},*)} \circ \theta \circ g^{n_1}|_\xi = \varphi_{n_1}|_{(\xi\times\{*\})},$$

where $(\star)$ is by (4.4.10) in Corollary 4.45. Thus, since $\varphi|_\xi$ is defined as the limit of the functions $\varphi_n|_\xi$, (4.6.19) follows. Moreover, (4.6.19) also implies that $\varphi(I_{(\underline{s},*)}) \subset \Gamma(\underline{s},*)$, since by (4.6.7), $\varphi_n(I_{(\underline{s},*)}) \subset \gamma^n_{(\underline{s},*)}$ for all $n \geq 0$.

To prove surjectivity of $\varphi|_{I_{(\underline{s},*)}}$, let us fix any $z \in \Gamma(\underline{s},*) \cap I(f)$. By Proposition 4.8, there exists $M \in \mathbb{N}$ such that $z \in \gamma^M_{(\underline{s},*)}$, and hence $f^M(z) \in \gamma^0_{(\sigma^M(\underline{s}),*)}$. Recall that $\gamma^0_{\sigma^M(\underline{s})} := \theta(\pi(J_{\sigma^M(\underline{s})}))$. By Theorem 4.31, there exists $p \in J_{\sigma^M(\underline{s})} \cap J_R(g)$ such that $\pi(w) = p$ for all points $w \in J_{\sigma^M(\underline{s})}$ with lower potential than p, and $\pi(w) = w$ otherwise. Hence, if β is the subcurve in $J_{\sigma^M(\underline{s})}$ that includes p and all points with greater potential, by this and Corollary 4.45, φ_0 maps $\beta \times \{*\}$ bijectively to $\gamma^0_{\sigma^M(\underline{s})}$. Since by Proposition 4.27 $J_{\underline{s}}$ is mapped bijectively to $J_{\sigma^M(\underline{s})}$, by (4.6.6), if $\delta := g^{-n}(J_{\sigma^M(\underline{s})}) \cap J_{\underline{s}}$, then $\delta \times \{*\}$ is mapped bijectively to $\gamma^0_{\sigma^M(\underline{s})}$ under φ_M. In particular, there exists $w \in \delta$ such that $\varphi_M((w,*)) = z$. Arguing as previously when showing (4.6.19), it must occur that $\varphi|_{(\delta\times\{*\})} \equiv \varphi_M|_{(\delta\times\{*\})}$, and hence surjectivity follows. △

The equality $\varphi(I(g)_\pm) = I(f)$ is now a consequence of the claim together with Proposition 4.8 and (4.6.17). In addition, for each $(\underline{s},*) \in \operatorname{Addr}(g)_\pm$, if $(e_{\underline{s}},*) \in J(g)_\pm \setminus I(g)_\pm$, then by (4.6.17) and (4.6.18), $\varphi((e,*)) \in J(f) \setminus I(f)$, and so by the previous claim and continuity of φ, $\varphi|_{J_{(\underline{s},*)}}$ is injective and

$$\Gamma(\underline{s},*) \subset \varphi(J_{(\underline{s},*)}) \subset \overline{\Gamma}(\underline{s},*). \tag{4.6.21}$$

In order to prove surjectivity of φ, let $J(g)_\pm \cup \{\tilde{\infty}\}$ be the one point compactification of $J(g)_\pm$ provided by Lemma 4.58, and denote by $J(f) \cup \{\infty\}$ the compactification of $J(f)$ as a subset of the Riemann sphere $\widehat{\mathbb{C}}$. By Lemma 4.58 and (4.6.16), given a sequence $\{x_n\}_{n\in\mathbb{N}} \subset J(g)_\pm \cup \{\tilde{\infty}\}$, we have

$$\lim_{n\to\infty} x_n = \tilde{\infty} \quad\Longleftrightarrow\quad \lim_{n\to\infty} \operatorname{Proj}(x_n) = \infty \quad\Longleftrightarrow\quad \lim_{n\to\infty} \varphi(x_n) = \infty. \tag{4.6.22}$$

Since by Lemma 4.58 $J(g)_\pm \cup \{\tilde{\infty}\}$ is a sequential space, and so is $\widehat{\mathbb{C}}$, the notions of continuity and sequential continuity for functions between these spaces are equivalent. Therefore, by (4.6.22), we can extend φ to a continuous map $\hat{\varphi} : J(g)_\pm \cup \{\tilde{\infty}\} \to J(f) \cup \{\infty\}$ by defining $\hat{\varphi}(\tilde{\infty}) = \infty$. By continuity of $\hat{\varphi}$, we have that $\hat{\varphi}\left(J(g)_\pm \cup \{\tilde{\infty}\}\right)$ is compact. By definition of $\hat{\varphi}$, it must be the case that $\hat{\varphi}(J(g)_\pm) = \varphi(J(g)_\pm)$, and by removing $\{\infty\}$ from the codomain of $\hat{\varphi}$, we can conclude that $\varphi(J(g)_\pm)$ is (relatively) closed in $J(f)$ with respect to the original topologies. By this and using Theorem 2.17,

$$I(f) = \varphi(I(g)_\pm) \subset \varphi(J(g)_\pm) \subset J(f) = \overline{I(f)},$$

and so $\varphi(J(g)_\pm)$ must be equal to $J(f)$, showing that φ is surjective. Moreover, arguing exactly the same way, we can see that for each $(\underline{s},*) \in \operatorname{Addr}(f)_\pm$, the set $\varphi(J_{(\underline{s},*)})$ is closed in $J(f)$, and hence, by (4.6.21), $\varphi\colon J_{(\underline{s},*)} \to \overline{\Gamma}(\underline{s},*)$ is a bijection. In particular, $\overline{\Gamma}(\underline{s},*)$ is a canonical ray together with its endpoint.

Finally, by Observation 4.12, each $z \in \mathcal{S} \supset I(f)$ is contained in $\#\operatorname{Addr}(z) = \prod_{j=0}^{\infty} \deg(f, f^j(z))$ Γ-curves. By the claim in this proof, for each $z \in I(f)$, $\#\varphi^{-1}(z) = \operatorname{Addr}(z)$. Moreover, since f is strongly postcritically separated, by items (b) and (b) in Definition 3.3, there exist constants $N, c \in \mathbb{N}$ such that for each $z \in J(f)$, $\#(\operatorname{Orb}^+(z) \cap \operatorname{Crit}(f)) \leq c$ and $\deg(f,w) \leq N$ for all $w \in \operatorname{Crit}(f)$. Hence, letting $K := N^c$ the claim in the statement follows. □

Proof of Theorem 1.8. Is a direct consequence of Theorem 4.64. □

Proof of Corollary 1.9. Note that f satisfying the hypotheses of Theorem 1.8 is in particular criniferous (Corollary 4.50) and does not have asymptotic values on its Julia set. Hence, by Observation 4.14, proving that all canonical rays for f land suffices to conclude that all its dynamic rays land. Moreover, by the same observation and since a dynamic ray is defined as the maximal injective curve among those satisfying the properties in Definition 1.5, each canonical ray must be contained in $\mathit{\Gamma}(\underline{s}, *)$ for some signed address $(\underline{s}, *) \in \operatorname{Addr}(f)_{\pm}$, and conversely each $\mathit{\Gamma}(\underline{s}, *)$ contains at most one canonical ray. Since by Theorem 4.64, for each $(\underline{s}, *) \in \operatorname{Addr}(f)_{\pm}$, $\varphi(J_{(\underline{s}, *)}) = \overline{\mathit{\Gamma}}(\underline{s}, *)$ is a canonical ray together with its landing point, the corollary follows. $\square$

Chapter 5

Further results and questions

5.1 Topological models for cosine dynamics

This section concerns the dynamics of cosine maps, that is, those of the form $z \mapsto ae^z + be^{-z}$ for $a, b \in \mathbb{C}^*$. More specifically, we provide a *simpler* and more *explicit* model for the action of strongly postcritically separated cosine maps on their Julia sets, that we use to draw conclusions on the landing behaviour of their dynamic rays. This model is inspired by the model constructed by Rempe-Gillen for exponential maps $E_\kappa \colon z \mapsto e^z + \kappa$. More specifically, he constructs a model for their dynamics in suitable subsets of their escaping sets [Rem06, Theorem 4.2], following previous work in this direction [AO93, BDD$^+$01]. Some of the advantages that Rempe-Gillen's model presents over previous models for the exponential family is that its dynamics are easy to analyse and it is defined without referring to a specific exponential map, but instead relates to E_κ for all $\kappa \in \mathbb{C}$. In addition, the model is conjugate to the whole Julia set of those exponential maps with attracting or parabolic parameter [Rem06, Theorem 1.2]. This provides a combinatorial framework for exponential maps that in particular allows to draw further conclusions on their topological dynamics. For example, it is used in [ARG17] to determine the topology of escaping endpoints of certain exponentials.

Given that cosine maps act like the exponential map, up to a constant factor, in left and right half-planes sufficiently far away from the imaginary axis, it is expected that a similar model exists for their dynamics. In fact this is the case, as we shall show in this section. This was already claimed in [MB12, Appendix A], where the construction of a model for the map $\pi \sinh$ is sketched. We also note that even if written in different terms, the existence of a model for the escaping set of cosine maps can be expected from the work of Rottenfußer and Schleicher [RS08] on the dynamics of cosine maps in their escaping sets, see also [Sch07]. We indeed will make use of some of the estimates appearing in [RS08] for our construction. More specifically, we start by defining a model for cosine dynamics, and conjugate this model in Theorem 5.21 to any disjoint type cosine map on its Julia set. Combining this result with a conjugacy near infinity between any cosine map and a disjoint type map on its parameter space [Rem09], we can relate this model to the dynamics near infinity of any cosine map, see Corollary 5.23. Then, we consider two copies of the model and construct a new model and prove a more detailed version of Theorem 1.10. Finally, we focus on the dynamics of the maps cosh and $\cosh^2$, providing an explicit combinatorial description of the overlappings occurring between their canonical tails, and concluding that no two of their dynamic rays land together, see Proposition 5.34.

5.1 (Basic properties of cosine maps)**.** Each cosine map $f(z) := ae^z + be^{-z}$ with $a, b \in \mathbb{C}^*$ is $2\pi i$-periodic and has exactly two critical values, namely $\pm 2\sqrt{ab}$. Furthermore, any preimage of a critical value is a critical point of local degree 2, and hence both critical values are *totally ramified*. More specifically,

$$\operatorname{Crit}(f) = \left\{ \frac{1}{2} \ln\left(\frac{a}{b}\right) + \pi i n : n \in \mathbb{Z} \right\},$$

where the branch of the logarithm is chosen such that $|\operatorname{Im}(\frac{1}{2}\ln(\frac{a}{b}))| \leq \pi/2$. It is easy to check that f has no asymptotic values, and thus, $S(f) =: \{v_1, v_2\}$, with $v_i = \pm 2\sqrt{ab}$ and choosing signs so that v_1 is the image of $\frac{1}{2}\ln(\frac{a}{b}) + 2\pi i \mathbb{Z}$, while v_2 is the image of $\frac{1}{2}\ln(\frac{a}{b}) + \pi i \mathbb{Z}$. In particular, $f \in \mathcal{B}$, and since in addition f is of order one, $f \in \mathcal{CB}$, see Proposition 4.39. Moreover, by Denjoy-Carleman-Ahlfors Theorem, the number of tracts of f, defined for any choice of bounded domain $D \supset S(f)$, is at most two. Note that for any such domain D, f maps points for which the absolute value of their real part is sufficiently large, to $\mathbb{C} \setminus D$. Hence, a left and a right half plane are contained in the set of tracts, which implies that f has at least two, and hence exactly two, tracts.

Since we have seen that any cosine map belongs to $\mathcal{CB}$, in particular any disjoint type cosine map has a Julia Cantor bouquet. Moreover, they all belong to the same parameter space in the sense of §4.4:

Observation 5.2 (Parameter space of cosine maps)**.** All cosine maps belong to the same parameter space. That is, any two cosine maps are quasiconformally equivalent. To see this, let $f(z) := ae^z + be^{-z}$ and $g(z) := ce^z + de^{-z}$ for $a, b, c, d \in \mathbb{C}^*$. Consider the quasiconformal maps $\psi(z) := z + \log\sqrt{\frac{bc}{ad}}$ and $\varphi(z) := \sqrt{\frac{bc}{ad}}z$. Then, for all $z \in \mathbb{C}$,

$$(f \circ \psi)(z) = ae^z\sqrt{\frac{bc}{ad}} + be^{-z}\sqrt{\frac{ad}{bc}} = ce^z\sqrt{\frac{ab}{cd}} + de^{-z}\sqrt{\frac{ab}{cd}} = (\varphi \circ g)(z).$$

Consequently, by Corollary 4.37, constructing a topological model for the dynamics of any specific disjoint type cosine map suffices to obtain a model for any cosine disjoint type map. We could have followed this approach, but for the sake of generality, we define a model without referring to a specific cosine map, and we conjugate it to each disjoint type cosine map. As pointed out before in this thesis, the map

$$\cosh(z) := \frac{e^z + e^{-z}}{2}$$

is an example of strongly postcritically separated cosine map, see 5.3 below. Thus, in view of Observation 5.2, we suggest the reader to keep in mind the specific combinatorics the family of maps $\lambda\cosh$ for $\lambda \in \mathbb{R}^+$ whenever we deal with general cosine maps in this section.

5.3 (Dynamics within the one-parameter family $\lambda\cosh$)**.** Let us consider the family $g_\lambda(z) := \lambda\cosh(z)$ with $\lambda \in \mathbb{R}^+$. For each $\lambda \in \mathbb{R}$, $S(g_\lambda) = \operatorname{CV}(g_\lambda) = \{-\lambda, +\lambda\}$, and the restriction $g_\lambda|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is a real even function such that $\min_{x\in\mathbb{R}} |g_\lambda(x)| = |g_\lambda(0)| = \lambda$ and $\mathbb{R} \subset I(g_\lambda)$ for λ large enough. Moreover, $g_\lambda(i\mathbb{R}) \subset [-\lambda, \lambda]$ and in particular, $\operatorname{Crit}(g_\lambda) = \{\pm n\pi i : n \in \mathbb{Z}\}$. In addition, since $g_\lambda(\lambda) = g_\lambda(-\lambda)$, we have that $P(g_\lambda) = \operatorname{Orb}^+(\lambda) \cup \{-\lambda\}$, and the orbit of λ consists of a sequence of points of increasing moduli that escape to infinity at an exponential rate. Hence, for every $\lambda \in \mathbb{R}$, g_λ is a strongly postcritically separated function in class $\mathcal{B}$ and such that $J(g_\lambda) = \mathbb{C}$. For $\lambda < 1/2$, there exists an attracting fixed point in $[-\lambda + \epsilon, \lambda + \epsilon]$ for some small $\epsilon > 0$, and the subinterval $[-\lambda + \epsilon, \lambda + \epsilon]$ of the real axis is mapped into itself, and so, belongs to the immediate basin of attraction of the fixed point. Hence, for any $\lambda < 1/2$, g_λ is of disjoint type, see Proposition 2.20.

Recall from 5.1 that each cosine map g has two tracts for any choice of $D \Supset S(g)$ on their definition. In order to guarantee expansion within tracts, we fix a pair of them whose boundaries are sufficiently far from the imaginary axis:

Definition 5.4 (Normalized disjoint type cosine maps)**.** For each cosine function $g(z) := ae^z + be^{-z}$ with $a, b \in \mathbb{C}^*$, denote

$$\mathcal{L}(g) := \max\left\{\left(\sqrt{\left|\frac{2b}{a}\right|} + \sqrt{\left|\frac{2a}{b}\right|}\right)(|a| + |b|), 8|ab|, 1, \frac{1}{2}\ln\left|\frac{2b}{a}\right|, \frac{1}{2}\ln\left|\frac{2a}{b}\right|, \ln\frac{16}{|ab|}\right\}.$$

If in addition g is of disjoint type, then we say that g is *normalized* if there exists a pair of tracts $\mathcal{T}_g$ for g such that $\mathcal{T}_g \subseteq \{z : |\operatorname{Re}(z)| > \mathcal{L}(g)\}$ and in addition $S(g) \subset \mathbb{D}_{\mathcal{L}(g)} \subset \mathbb{C} \setminus g(\mathcal{T}_g)$. We then say that $\mathcal{T}_g$ are *expansion* tracts.

Observation 5.5 (Euclidean expansion for normalized maps)**.** A simple calculation shows that if g is a cosine map, then $|g'(z)| > 2$ for all $\{z : |\mathrm{Re}(z)| > \mathcal{L}(g)\}$. See [RS08, Lemma 3.6]. Moreover, recall that if g is of disjoint type, then $J(g) = \bigcap_{k\geq 0} g^{-k}(\mathcal{T}_g)$ and $g^{-n}(\mathcal{T}_g) \subset \mathcal{T}_g$ for all $n \geq 0$.

We note that normalized disjoint type cosine maps exist, and in fact there are plenty of them:

Proposition 5.6 (Existence of normalized disjoint type maps)**.** *Let f be a cosine map. Then, for all $\lambda \in \mathbb{C}$ with $|\lambda|$ small enough, $g_\lambda := \lambda f$ is a disjoint type normalized map.*

Proof. By assumption, $f(z) := ae^z + be^{-z}$ for some $a, b \in \mathbb{C}^*$. For $\lambda \in \mathbb{C}$ with small enough modulus, by Proposition 4.35, λf is of disjoint type, $\mathcal{L}(\lambda f) \leq \mathcal{L}(f)$, and $S(\lambda f) = \{\pm 2\lambda\sqrt{ab}\} \subset \mathbb{D}_{\mathcal{L}(\lambda f)}$. Let us fix $R > \mathcal{L}(f)$ such that

$$V := \bigcup_{k\in\mathbb{Z}} \{z + 2\pi k i : z \in \mathbb{D}_R\} \supset \{z : |\operatorname{Re} z| \leq \mathcal{L}(f)\}, \tag{5.1.1}$$

and note that for any map $g := \lambda f$ with $|\lambda|$ small enough, $g(\overline{\mathbb{D}}_R) \subset \overline{\mathbb{D}}_{\mathcal{L}(f)}$. In particular, by $2\pi i$-periodicity of g, $g(V) \subset \mathbb{D}_{\mathcal{L}(f)}$. Then, by (5.1.1) it holds $\mathcal{T}_g := g^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}_R) \subset \{z : |\operatorname{Re} z| > \mathcal{L}(f)\}$, and in addition, since by construction $\mathbb{D}_{\mathcal{L}(g)} \subset \mathbb{D}_{\mathcal{L}(f)} \subset \mathbb{D}_R = \mathbb{C} \setminus g(\mathcal{T}_g)$, $\mathcal{T}_g$ are expansion tracts. $\square$

5.7 (Inverse branches for normalized disjoint type maps)**.** Let g be a normalized disjoint type cosine map given by $g(z) := ae^z + be^{-z}$ for some $a, b \in \mathbb{C}^*$, and let $\mathcal{T}_g$ be a pair of expansion tracts. Let $S(g) =: \{v_1, v_2\}$ with v_1 and v_2 labelled according to 5.1. Since g is normalized, $S(g) \subset D := \mathbb{C} \setminus g(\mathcal{T}_g)$ and $D \subset \mathbb{C} \setminus \mathcal{T}_g$. If $\mathrm{Im}(v_1) > \mathrm{Im}(v_2)$, we define δ as the vertical straight line starting at v_1 in upwards direction. If on the contrary $\mathrm{Im}(v_1) < \mathrm{Im}(v_2)$, δ will represent the restriction to $\mathbb{C} \setminus D$ of the downwards vertical line joining v_2 to infinity. In any case, $\delta \subset \mathbb{C} \setminus (\mathcal{T}_g \cup D)$, and so we can define fundamental domains for g as the connected components of $\mathcal{T}_g \setminus g^{-1}(\delta)$. Since g is in the cosine family, by definition, all points in $\mathbb{R}$ whose modulus is large enough belong to $I(g)$, and hence must be totally contained in a fundamental domain. By $2\pi i$-periodicity of g, the same occurs to all their $2\pi i$ translates. Hence, for each $n \in \mathbb{Z}$, we denote by $F_{(n,R)}$ the fundamental domain that contains an unbounded subset of $2\pi n i\mathbb{R}^+$, and by $F_{(n,L)}$ the fundamental domain that contains an unbounded component of $2\pi n i\mathbb{R}^-$. Since g maps each fundamental domain to its image $g(\mathcal{T}_g) \setminus \delta$ as a conformal isomorphism, see Proposition 2.19(3), we can define for each $(n, *) \in (\mathbb{Z} \times \{L, R\})$ the inverse branch

$$g^{-1}_{(n,*)} \colon g(\mathcal{T}_g) \setminus \delta \to F_{(n,*)}, \tag{5.1.2}$$

which in particular is a bijection.

Observation 5.8 (Horizontal straight lines contained in fundamental domains)**.** Following 5.7, by construction, there is a constant $A > \mathcal{L}(g)$ so that for all $n \in \mathbb{Z}$,

$$\{z : \operatorname{Re} z < -A \text{ and } \operatorname{Im} z = 2\pi n\} \subset F_{(n,L)} \qquad \text{and}$$
$$\{z : \operatorname{Re} z > A \text{ and } \operatorname{Im} z = 2\pi n\} \subset F_{(n,R)}.$$

We note that our choice of fundamental domains in 5.7 agrees with the partition defined in [RS08, Sections 1 and 2], where the maps "$g^{-1}_{(n,*)}$" are labelled as "L_s". Then, the estimates appearing in [RS08] regarding this partition and inverse branches apply to our setting. In particular, we will use the following:

Proposition 5.9 (Properties of the partition [RS08, Lemmas 2.3 and 3.4])**.** *In the setting described in 5.7, the following hold:*

- *If $z, w \in F_{(n,*)}$ for some $(n,*) \in (\mathbb{Z} \times \{L,R\})$, then $|\operatorname{Im} z - \operatorname{Im} w| < 3\pi$ and $|\operatorname{Im} z - 2\pi n| < 3\pi$.*
- *If $w \in g(\mathcal{T}_g) \setminus \delta$, then for each $(n,*) \in (\mathbb{Z} \times \{L,R\})$ there exists $r^\star \in \mathbb{C}$ with $|r^\star| < 1$ and such that*

$$g^{-1}_{(n,*)}(w) := \begin{cases} \ln(w) - \log a + 2\pi i n + r^\star & \text{if} \quad * = R \\ -\ln(w) + \log b + 2\pi i n + r^\star & \text{if} \quad * = L. \end{cases}$$

5.10 (External addresses for normalized functions)**.** For each disjoint type normalized g, we define external addresses for g using the fundamental domains specified in 5.7. In particular, by Observation 5.5, each point in $J(g)$ belongs to some Julia constituent of g, see (2.4.4). Recall that we denote the set of admissible addresses for g as $\operatorname{Addr}(g)$. In particular, we endow $\operatorname{Addr}(g)$ with the cyclic order topology specified in 2.31.

Notation. For each element $(n,*) \in (\mathbb{Z} \times \{L,R\})$, we denote $|(n,*)| := |n|$ and $\{(n,*)\} := n$.

A model for disjoint type cosine maps

In this subsection we construct the promised topological model for cosine dynamics and conjugate it to any disjoint cosine map on its Julia set.

5.11 (Topological space $(\mathcal{M}, \tau_{\mathcal{M}})$)**.** Consider the set

$$\mathcal{M} := [0,\infty) \times (\mathbb{Z} \times \{L,R\})^{\mathbb{N}}.$$

Let "$<_{\mathbb{Z}}$" be the usual linear order on integers. We define a total order in the set $(\mathbb{Z} \times \{L, R\})$ as follows:

$$(n,*) < (m,\star) \iff \begin{cases} * = R = \star & \text{and} \quad n <_{\mathbb{Z}} m, & \text{or} \\ * = L = \star & \text{and} \quad m <_{\mathbb{Z}} n, & \text{or} \\ * = L & \text{and} \quad \star = R, \end{cases} \tag{5.1.3}$$

that induces a lexicographic order "$<_\ell$" in $(\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$. In turn, we define a cyclic order induced by $<_\ell$ in the usual way: for $\underline{s}, \underline{\alpha}, \underline{\tau} \in (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$,

$$[\underline{s}, \underline{\alpha}, \underline{\tau}]_\ell \quad \text{if and only if} \quad \underline{s} <_\ell \underline{\alpha} <_\ell \underline{\tau} \quad \text{or} \quad \underline{\alpha} <_\ell \underline{\tau} <_\ell \underline{s} \quad \text{or} \quad \underline{\tau} <_\ell \underline{s} <_\ell \underline{\alpha}.$$

Moreover, given two different elements $\underline{s}, \underline{\tau} \in (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$, we define the *open interval* from $\underline{s}$ to $\underline{\tau}$, denoted by $(\underline{s}, \underline{\tau})$, as the set of all points $x \in (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ such that $[\underline{s}, x, \underline{\tau}]$. The collection of all such open intervals forms a base for the cyclic order topology. We then provide the space $\mathcal{M}$ with the topology $\tau_{\mathcal{M}}$ defined as the product topology of $[0, \infty)$ with the usual topology, and $(\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ with the just described cyclic order topology.

Notation. If for some $k \geq 0$, $\underline{s} = s_0 s_1 s_2 \ldots \in (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ is such that $s_j = s_k$ for all $j > k$, then we write $\underline{s} = s_0 s_1 \ldots \overline{s_k}$.

Observation 5.12 (Correspondence between topological spaces)**.** Let g be any normalized cosine disjoint type map, and suppose that $\operatorname{Addr}(g)$ has been defined following 5.10. In particular, $\operatorname{Addr}(g)$ is endowed with a cyclic order topology following 2.31. Then, there exists a one-to-one correspondence between $(\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ and $\operatorname{Addr}(g)$ that preserves their topologies. Namely, the one that converts sequences as

$$(m, \star)(n, *) \ldots \leftrightsquigarrow F_{(m,\star)} F_{(n,*)} \ldots .$$

Then, since the curve δ chosen in 5.7 is a vertical straight line, the linear order in fundamental domains chosen to define the cyclic order topology in $\operatorname{Addr}(g)$, see (2.4.7), agrees with the linear order (5.1.3) that determines the topology in $(\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$, up to the specified correspondence. Hence, from now on we omit the specification of the correspondence, and $\underline{s}$ might denote either an element of $(\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ or its corresponding element on $\operatorname{Addr}(g)$.

Definition 5.13 (A topological model for cosine dynamics)**.** Let $(\mathcal{M}, \tau_{\mathcal{M}})$ as specified in 5.11. Define $\mathcal{F}\colon (\mathcal{M},\tau_{\mathcal{M}}) \to (\mathcal{M},\tau_{\mathcal{M}})$ as

$$\mathcal{F}(t,\underline{s}) := (F(t) - 2\pi|s_1|, \sigma(\underline{s})),$$

where σ denotes the shift map on one-sided infinite sequences of $(\mathbb{Z}\times\{L,R\})^{\mathbb{N}}$ and $F(t) := e^t - 1$ is the standard map that codes exponential growth. Let $T\colon \mathcal{M} \to [0,\infty)$ given by $T(t,\underline{s}) := t$ be the projection of a point into the first coordinate. We set

$$J(\mathcal{F}) := \{x \in \mathcal{M}\colon\ T(\mathcal{F}^n(x)) \geq 0 \text{ for all } n \geq 0\}, \qquad \text{and}$$
$$I(\mathcal{F}) := \{x \in J(\mathcal{F})\colon T(\mathcal{F}^n(x)) \to \infty \text{ as } n \to \infty\}.$$

We say that $\underline{s} \in (\mathbb{Z}\times\{L,R\})^{\mathbb{N}}$ is *exponentially bounded* if $(t,\underline{s}) \in J(\mathcal{F})$ for some $t > 0$. We moreover let

$$t_{\underline{s}} := \begin{cases} \min\{t \geq 0 : (t,\underline{s}) \in J(\mathcal{F})\} & \text{if } \underline{s} \text{ is exponentially bounded,} \\ \infty & \text{otherwise.} \end{cases}$$

That is, $J(\mathcal{F})$ is the set of all points that stay in the space $\mathcal{M}$ under iteration of $\mathcal{F}^n$ for all $n \geq 0$.

Observation 5.14 (Relation between cosine and exponential model)**.** Let $\mathcal{M}_{\exp} := [0,\infty)\times\mathbb{Z}^{\mathbb{N}}$ be a space with the product topology, and define the map $\mathcal{F}_{\exp}\colon \mathcal{M}_{\exp} \to \mathcal{M}_{\exp}$ and the set $J(\mathcal{F}_{\exp})$ by replacing in Definition 5.13 the space $\mathcal{M}$ with $\mathcal{M}_{\exp}$. Then, $(\mathcal{F}, J(\mathcal{F}_{\exp}))$ is the model for the dynamics of exponential maps described in [Rem06, Section 3] and [ARG17, Definition 3.1]. We note that there does not exist an order preserving bijection from $\mathbb{Z}^{\mathbb{N}}$ with the usual lexicographic order and $((\mathbb{Z}\times\{L,R\})^{\mathbb{N}}, <_{\ell})$, and hence the models are not the same. This is expected, since exponential maps have a single tract contained on a right half plane, while cosine maps have two tracts, as described in 5.1. However, the spaces $\mathcal{M}$ and $\mathcal{M}_{\exp}\times\{L,R\}^{\mathbb{N}}$ with the product topology are homeomorphic via the map $h\colon \mathcal{M}_{\exp}\times\{L,R\}^{\mathbb{N}} \to \mathcal{M}$ given by $h(t,\underline{s},\underline{\omega}) := (t,(s_0,w_0)(s_1,w_1)(s_2,w_2)\ldots)$, where $\underline{s} = s_0s_1\ldots \in \mathbb{Z}^N$ and $\underline{\omega} = w_0w_1w_2\ldots \in \{L,R\}^{\mathbb{N}}$. This can be seen recalling that a base for the product topology of $\mathcal{M}_{\exp}\times\{L,R\}^{\mathbb{N}}$ is given by cylinders, and the image of each such cylinder under h can be expressed as a union of intervals of the cyclic order topology, and conversely, the preimage of open intervals in $\tau_{\mathcal{M}}$ are a union of cylinders of the first space. In particular, $J(\mathcal{F})$ is homeomorphic to $J(\mathcal{F}_{\exp})\times\{L,R\}^{\mathbb{N}}$, where each subspace has the topology respectively induced from $\mathcal{M}$ and $\mathcal{M}_{\exp}\times\{L,R\}^{\mathbb{N}}$.

We shall use the relation specified above between the exponential and cosine models to prove properties of the latter:

Proposition 5.15 (Properties of the cosine model)**.** *The space $J(\mathcal{F})$ with the induced subspace topology from $(\mathcal{M}, \tau_{\mathcal{M}})$ admits the* 1*-point compactification, and the resulting space $J(\mathcal{F}) \cup \{\tilde{\infty}\}$ is a sequential space. Moreover, $\mathcal{F}|_{J(\mathcal{F})}$ is continuous.*

Proof. By Observation 5.14, $J(\mathcal{F})$ is homeomorphic to $J(\mathcal{F}_{\text{exp}}) \times \{L, R\}^{\mathbb{N}}$. In turn, $J(\mathcal{F}_{\text{exp}})$ is homeomorphic to a straight brush, which is a subset of $\mathbb{R}^2$ with the usual Euclidean metric, see [ARG17, Theorem 3.3], and $\{L, R\}^{\mathbb{N}}$ is homeomorphic to the Cantor set. Hence, $J(\mathcal{F}_{\text{exp}}) \times \{L, R\}^{\mathbb{N}}$ is a locally compact, Hausdorff and second-countable space. Thus, it admits the one-point compactification and the resulting space is first countable, and so sequential. Consequently, the same holds for $J(\mathcal{F})$ and its compactification.

In order to prove continuity of $\mathcal{F}|_{J(\mathcal{F})}$, let us fix an arbitrary $(t, \underline{s}) \in J(\mathcal{F})$ and let V be an open neighbourhood of $\mathcal{F}(t, \underline{s})$. Without loss of generality, we may assume that $V = ((t_1, t_2) \times J) \cap J(F)$ for some open interval $J \in (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ and $t_1, t_2 \in \mathbb{R}^+$ so that $t_1 \leq T(\mathcal{F}(t, \underline{s})) \leq t_2$. Suppose that $\underline{s} = s_0 s_1 \dots$ and denote $\tilde{J} := \{s_0 \underline{\tau} : \underline{\tau} \in J\}$. In particular, $\underline{s} \in \tilde{J}$, and since by definition of $\mathcal{F}$, $t = \log(T(\mathcal{F}(t, \underline{s}) + 1 + 2\pi\{s_1\})$ and the function log is increasing,

$$U := (\log(t_1 + 1 + 2\pi\{s_1\}), \log(t_2 + 1 + 2\pi\{s_1\})) \times \tilde{J}) \cap J(\mathcal{F})$$

is an open neighbourhood of $(t, \underline{s})$ such that $\mathcal{F}(U) \subset V$. □

Our first goal is, for each disjoint type cosine map g, to find a continuous map $\Phi : J(\mathcal{F}) \to J(g)$ that conjugates the dynamics of $\mathcal{F}$ to those of g in $J(g)$. In particular, the map Φ will send each point $(t, \underline{s}) \in J(\mathcal{F})$ to a point $z \in J(g)$ such that $z \in J_{\underline{s}}$ and $|\operatorname{Re} z| \approx t$, see Observation 5.12. Following the same strategy as we did in Chapter 4, we will obtain the map Φ as the limit of a series of *approximations* $\{\Phi_n\}_{n \in \mathbb{N}}$. The first approximation should be a projection from the space $J(\mathcal{F})$ to the dynamical plane of g.

Definition 5.16 (Projection function)**.** For each $A \geq 0$, we define the *projection function* $\mathcal{C}_A : J(\mathcal{F}) \to \mathbb{C}$ as

$$\mathcal{C}_A(t, \underline{s}) := \begin{cases} t + A + 2\pi\{s_0\}i & \text{if} \quad s_0 = (n, R) \text{ for some } n \in \mathbb{Z}, \\ -t - A + 2\pi\{s_0\}i & \text{otherwise,} \end{cases}$$

where $\underline{s} = s_0 s_1 \ldots$ and we recall that if $s_0 = (n, *)$, then $\{(n, *)\} = n$.

Observation 5.17 (The projection of $J(\mathcal{F})$ lies in fundamental domains)**.** Suppose that g is a disjoint type normalized function for which fundamental domains have been defined following 5.7. If A is the constant provided in Observation 5.8, then $\mathcal{C}_A(J(\mathcal{F}))$ is totally contained in the union of fundamental domains. More specifically, for each $(t, \underline{s}) \in J(\mathcal{F})$, if $\underline{s} = s_0 s_1 \ldots$, then $\mathcal{C}_A(t, \underline{s}) \subset F_{s_0}$, see also Observation 5.12.

Remark. The reason why instead of *projecting* each point $(t, \underline{s}) \in J(\mathcal{F})$ to a point of real part $\pm t$, but rather $\pm t \pm A$ for some constant A, is to ensure that for a fixed function g, the image of each $(t, \underline{s}) \in J(\mathcal{F})$ under a projection map lies in a fundamental domain of g, on which by Proposition 5.9 g *expands* the Euclidean metric. Note that unlike when constructing the semiconjugacy from Theorem 4.64 in §4.6, where $\varphi_0(J(g)_\pm) \subset J(f)$, now $\mathcal{C}_A(J(\mathcal{F})) \nsubseteq J(g)$. Nonetheless, since the limit function will be obtained as the limit of a composition of inverse branches whose images lie in $\mathcal{T}_g$, by Observation 5.5, its codomain will be $J(g)$.

Recall that cosine maps behave like the exponential map for points with modulus large enough and sufficiently far from the imaginary axis, and in particular contained in their tracts. An essential characteristic of our model for cosine dynamics is that, as occurs for the exponential map, for each $(t, \underline{s}) \in J(\mathcal{F})$, $|\mathcal{C}_A(\mathcal{F}(\underline{s}, t))|$ is roughly the exponential of its real part, plus the constant A. More precisely:

Proposition 5.18 (Model acts similar to the exponential)**.** *If $x = (t, \underline{s}) \in J(\mathcal{F})$, then for each $A > 0$,*

$$\frac{F(t) + A}{\sqrt{2}} \leq |\mathcal{C}_A(\mathcal{F}(t, \underline{s}))| \leq F(t) + A. \tag{5.1.4}$$

Proof. Suppose that $\underline{s} = s_0 s_1 \ldots$ and let $b = 2\pi\{s_1\}$. Then

$$\begin{aligned} |\mathcal{C}_A(\mathcal{F}(t, \underline{s}))| &= |\pm(F(t) - b + A) + ib| = \sqrt{(F(t) + A - b)^2 + b^2} \\ &= \sqrt{(F(t) + A)^2 - 2(F(t) + A)b + 2b^2}. \end{aligned} \tag{5.1.5}$$

The second inequality in (5.1.4) follows from the assumption $T(\mathcal{F}(t, \underline{s})) \geq 0$, that is, $F(t) - b \geq 0$, because by (5.1.5),

$$|\mathcal{C}_A(\mathcal{F}(t, \underline{s}))| \leq \sqrt{(F(t) + A)^2} \iff -2(F(t)+A)b + 2b^2 \leq 0 \iff b \leq F(t) + A,$$

where we have used that $A, b, F(t) \geq 0$. For the first inequality in (5.1.4) we have

$$\begin{aligned}\sqrt{(F(t)+A)^2} \leq \sqrt{2}|\mathcal{C}_A(\mathcal{F}(t,\underline{s}))| &\iff (F(t)+A)^2 - 4(F(t)+A)b + 4b^2 \geq 0\\ &\iff (F(t)+A-2b)^2 \geq 0.\end{aligned}$$

□

The underlying idea in the construction of a map that conjugates $\mathcal{F}$ and any disjoint type map g is the same as when constructing the semiconjugacy of Theorem 4.64. That is, for each $n \geq 0$, the function $\Phi_n \colon J(\mathcal{F}) \to \mathbb{C}$ will be defined the following way: we iterate each point $x = (t, \underline{s}) \in J(\mathcal{F})$, with $\underline{s} = s_0 s_1 \dots s_n \dots$, under the model function $\mathcal{F}$ a number n of times. In particular, $\mathcal{F}^n(t, \underline{s}) = (t', \sigma^n(\underline{s}))$ for some $t' > 0$. Next, we *move* to the dynamical plane of g using the function $\mathcal{C}_A$ for some constant A big enough such that $(\mathcal{C}_A \circ \mathcal{F}^n)(t, \underline{s}) \in F_{s_n}$. Then, we use the composition of n inverse branches of g of the form (5.1.2) to obtain a point in F_{s_0}, that will be $\Phi_n(x)$. See Figure 13. Finally, we will use (Euclidean) expansion of g on its tracts to show that $\{\Phi_n\}_{n \geq 0}$ is convergent. We now formalize these ideas:

Definition 5.19 (Functions Φ_n)**.** Let g be a normalized disjoint type cosine map, and let A be a constant provided by Observation 5.8. Then, for each $n \geq 0$ we define the function $\Phi_n : J(\mathcal{F}) \to \mathbb{C}$ as

$$\Phi_0(x) := \mathcal{C}_A(x) \qquad\qquad \Phi_{n+1}(x) := g_{s_0}^{-1}(\Phi_n(\mathcal{F}(x)),$$

for $x = (t, \underline{s})$ and $\underline{s} = s_0 s_1 \dots$.

The function Φ_0 is clearly well-defined. In order to see that for all $n \geq 1$ the function Φ_n is also well-defined, fix $x = (t, \underline{s}) \in J(\mathcal{F})$ and suppose that $\underline{s} = s_0 s_1 \dots$. Then, expanding definitions

$$\Phi_n(x) = \left(g_{s_0}^{-1} \circ g_{s_1}^{-1} \circ \dots \circ g_{s_{n-1}}^{-1} \circ \mathcal{C}_A \circ \mathcal{F}^n\right)(x). \tag{5.1.6}$$

By Observations 5.17 and 5.5, the composition of the inverse branches $\{g_{s_i}\}_{i<n}$ is well-defined on $\mathcal{C}_A(\mathcal{F}^n(x)) \in F_{s_n}$. Moreover, by construction, for all $n \geq 0$,

$$\Phi_n \circ \mathcal{F} = g \circ \Phi_{n+1}. \tag{5.1.7}$$

Proposition 5.20 (Continuity of the functions Φ_n)**.** *For each $n \geq 0$, the function $\Phi_n : J(\mathcal{F}) \to \mathbb{C}$ is continuous.*

Proof. Let us fix an arbitrary $(t, \underline{s}) \in J(\mathcal{F})$ with $\underline{s} = s_0 s_1 \dots s_n \dots$ as well as some $\epsilon > 0$. To show that $\Phi_0 \equiv \mathcal{C}_A$ is continuous, let $\mathcal{I} \subset (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ be any

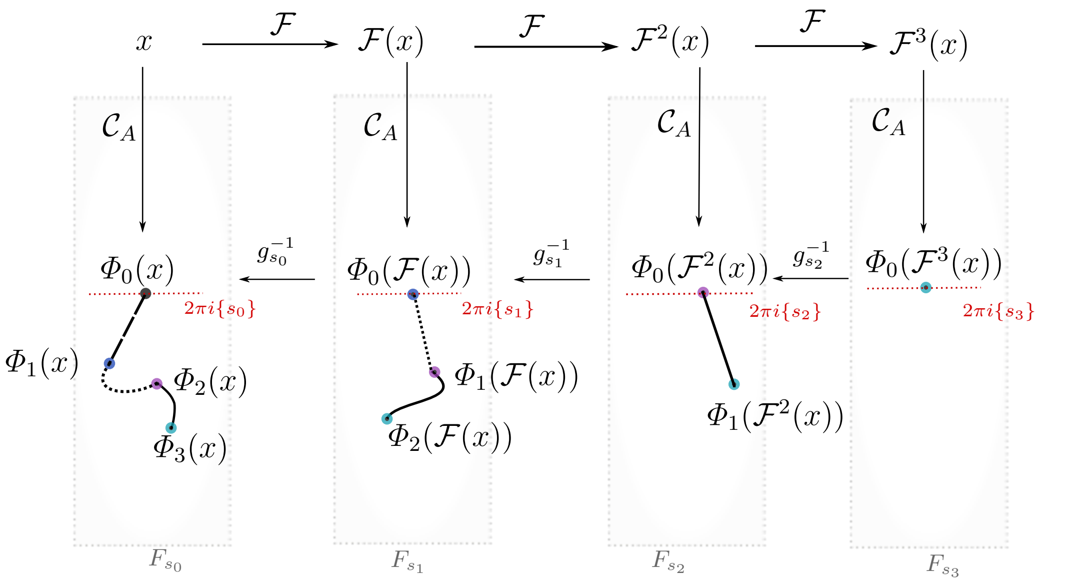

Figure 13: A schematic of the functions and curves involved in the definition of the functions $\{\Phi_n\}_{n\in\mathbb{N}}$.

open interval containing $\underline{s}$ and such that if $\underline{\tau} = \tau_0\tau_1\ldots \in \mathcal{I}$, then $s_0 = \tau_0$. Then, $U := ((t-\epsilon, t+\epsilon) \times \mathcal{I}) \cap J(\mathcal{F})$ is an open neighbourhood of $(t, \underline{s})$ such that

$$\mathcal{C}_A(U) \subset (\pm t - \epsilon, \pm t + \epsilon) \pm A + 2\pi i\{s_0\} \subset B_\epsilon(\pm t \pm A + 2\pi i\{s_0\}) = B_\epsilon(\mathcal{C}_A(t, \underline{s})),$$

where $\pm$ equals "$+$" or "$-$" depending on whether $s_0 = (n, R)$ or $s_0 = (n, L)$ for some $n \in \mathbb{Z}$. Hence, we have shown continuity of Φ_0. For each $n \geq 1$, let

$$L_n := g_{s_0}^{-1} \circ g_{s_1}^{-1} \circ \cdots \circ g_{s_{n-1}}^{-1} \circ \Phi_0 \circ \mathcal{F}^n$$

and note that for any subset $U \subset J(\mathcal{F})$ such that $\Phi_0(\mathcal{F}^n(U)) \subset F_{s_n}$, by Proposition 5.15 and the definition of the maps $\{g_{s_i}\}_{i<n}$, $L_n|_U$ is a continuous function, as it is a composition of continuous functions. By (5.1.6), $L_n(t, \underline{s}) = \Phi_n(t, \underline{s})$. Hence, in order to prove continuity of Φ_n at $(t, \underline{s})$, since by Observation 5.17 $\Phi_0(\mathcal{F}^n(t, \underline{s})) \subset F_{s_n}$, it suffices to find a neighbourhood $V \ni (t, \underline{s})$ such that $L_n|_V \equiv \Phi_n|_V$. Let $J_n \subset (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ be any open interval containing $\underline{s}$ and such that if $\underline{\tau} = \tau_0\tau_1\ldots\tau_n\ldots \in J_n$, then $s_i = \tau_i$ for all $0 \leq i \leq n$, and choose $t_1, t_2 \in \mathbb{R}^+$ so that $t_1 \leq t \leq t_2$. Then, $V := ((t_1, t_2) \times J_n) \cap J(\mathcal{F})$ satisfies the properties required and continuity of Φ_n follows. $\square$

Theorem 5.21 (Conjugacy between $\mathcal{F}$ and cosine maps of disjoint type)**.** *Let g be a disjoint type map in the cosine family. Then there exists a homeomorphism $\Phi : J(\mathcal{F}) \to J(g)$ such that $\Phi \circ \mathcal{F} = g \circ \Phi$. Moreover, $\Phi(I(\mathcal{F})) = I(g)$.*

Proof. We can assume without loss of generality that g is normalized, since by

Corollary 4.37 and Observation 5.2, proving this result for normalized functions is equivalent to proving it for all disjoint type cosine maps. Let $\{\Phi_n\}_{n\geq 0}$ be the sequence of functions from Definition 5.19 and suppose that $g(z) = ae^z + be^{-z}$ for some $a, b \in \mathbb{C}^*$ and all $z \in \mathbb{C}$. If $M := \max\{|a|, |b|\}$, then by Propositions 5.9 and 5.18, for each $x = (t, \underline{s}) \in J(\mathcal{F})$ with $\underline{s} = s_0 s_1 s_2 \ldots$,

$$\begin{aligned}|\mathrm{Re}(\Phi_1(x))| =& |\mathrm{Re}((g_{s_0}^{-1} \circ \mathcal{C}_A \circ \mathcal{F})(x))| \leq \ln|\mathcal{C}_A(\mathcal{F}(x))| + |\ln(M)| + 1 \\ \leq& t + A + |\ln(M)| + 2.\end{aligned}$$

Similarly, $|\mathrm{Re}(\Phi_1(x))| \geq t - \ln(\sqrt{2}) - |\ln(M)| - 2$. Since by definition of Φ_1 and Observation 5.8 both $\Phi_0(x)$ and $\Phi_1(x)$ lie in the same fundamental domain F_{s_0}, and thus both of them have either negative or positive real part simultaneously, using that $|\mathrm{Re}(\Phi_0(x))| = t + A$,

$$|\operatorname{Re}(\Phi_0(x)) - \operatorname{Re}(\Phi_1(x))| \leq A + \ln(\sqrt{2}) + |\ln(M)| + 2. \tag{5.1.8}$$

Moreover, by (5.1.8) and Proposition 5.9,

$$|\Phi_0(x) - \Phi_1(x)| \leq A + \ln(\sqrt{2}) + |\ln(M)| + 2 + 3\pi =: \mu, \tag{5.1.9}$$

where we note that the constant μ does not depend on the point x. Let $\mathcal{T}_g$ be a pair of expansion tracts for g. In particular, $\Phi_0(x)$ and $\Phi_1(x)$ lie in the same tract, and hence the straight segment joining these two points is totally contained in a connected component of $\{z : |\operatorname{Re} z| > \mathcal{L}(g)\}$, which is a convex set. Moreover, by (5.1.6), if $\mathsf{g}_{\underline{s},n}^{-1} := g_{s_0}^{-1} \circ g_{s_1}^{-1} \circ \cdots \circ g_{s_{n-1}}^{-1}$ for some $n \geq 1$, then

$$\Phi_n(x) = (\mathsf{g}_{\underline{s},n}^{-1} \circ \Phi_0 \circ \mathcal{F}^n)(x) \quad \text{and} \quad \Phi_{n+1}(x) = (\mathsf{g}_{\underline{s},n}^{-1} \circ \Phi_1 \circ \mathcal{F}^n)(x).$$

Note that if γ is the straight segment connecting $\Phi_0(\mathcal{F}^n(x))$ and $\Phi_1(\mathcal{F}^n(x))$, then since the map $\mathsf{g}_{\underline{s},n}^{-1}$ is a bijection to its image and it is a composition of bijections, $\mathsf{g}_{\underline{s},n}^{-1}(\gamma)$ is a curve with endpoints $\Phi_n(x)$ and $\Phi_{n+1}(x)$. Thus, using (5.1.9) and Observation 5.5, arguing as in the proof of Corollary 3.13,

$$|\Phi_{n+1}(x) - \Phi_n(x)| \leq \frac{|\Phi_0(\mathcal{F}^n(x)) - \Phi_1(\mathcal{F}^n(x))|}{2^n} \leq \frac{\mu}{2^n}. \tag{5.1.10}$$

Hence, $\{\Phi_n\}_{n\geq 0}$ is a uniformly Cauchy sequence of continuous functions, and so they converge uniformly to a continuous limit function $\Phi : J(\mathcal{F}) \to \mathbb{C}$, that by (5.1.7) satisfies

$$\Phi \circ \mathcal{F} = g \circ \Phi. \tag{5.1.11}$$

Note that for each $x \in J(\mathcal{F})$, $\Phi(x)$ is the limit of the backward orbit of a point in $\mathcal{T}_g$, see (5.1.6). Hence, by Observation 5.5, $\Phi(x) \in J(g)$ and thus, $\Phi(J(\mathcal{F})) \subset J(g)$. Moreover, since $\mathcal{C}_A \equiv \Phi_0$, for each $x \in J(\mathcal{F})$,

$$|\Phi(x) - \mathcal{C}_A(x)| \leq \sum_{n=0}^{\infty} |\Phi_{n+1}(x) - \Phi_n(x)| \leq \sum_{j=0}^{\infty} \frac{\mu}{2^n} = 2\mu. \tag{5.1.12}$$

This means for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset J(\mathcal{F})$ that $\Phi(x_n) \to \infty$ if and only if $\mathcal{C}_A(x_n) \to \infty$ as $n \to \infty$. By this, the definition of $I(\mathcal{F})$ and Proposition 5.18,

$$\begin{aligned} x \in I(\mathcal{F}) &\Leftrightarrow \lim_{n\to\infty} T(\mathcal{F}^n(x)) = \infty \Leftrightarrow \lim_{n\to\infty} |\mathcal{C}_A(\mathcal{F}^n(x))| = \infty \\ &\Leftrightarrow \lim_{n\to\infty} \Phi(\mathcal{F}^n(x)) = \lim_{n\to\infty} g^n(\Phi(x)) = \infty \Leftrightarrow \Phi(x) \in I(g). \end{aligned} \tag{5.1.13}$$

Equivalently, $\Phi(I(\mathcal{F})) \subseteq I(g)$ and $\Phi(J(\mathcal{F}) \setminus I(\mathcal{F})) \subseteq J(g) \setminus I(g)$. Consequently, surjectivity of Φ would imply $\Phi(I(\mathcal{F})) = I(g)$.

Claim. The function $\Phi : J(\mathcal{F}) \to J(g)$ is surjective.

Proof of claim. Fix an arbitrary $z \in J(g)$. Then, by (2.4.4), $z \in J_{\underline{s}}$ for some $\underline{s} = s_0 s_1 s_2 \ldots \in \operatorname{Addr}(g)$, where external addresses have been defined for g following 5.10. Note that by definition, the function $\mathcal{F}$ is injective on its first coordinate, that is, for each fixed $\underline{s} \in (\mathbb{Z} \times \{-,+\})^{\mathbb{N}}$, $\mathcal{F}_{\underline{s}} := \mathcal{F}(\cdot, \underline{s}) \colon \mathbb{R}^+ \to \mathbb{C}$ given by $t \mapsto \mathcal{F}(t, \underline{s})$ is injective. Hence, we can consider the sequence of real positive numbers $\{t_k\}_{k\geq 0}$ uniquely determined by the equations

$$\mathcal{F}^k(t_k, \underline{s}) = (|\operatorname{Re}(g^k(z))|, \sigma^k(\underline{s})).$$

In particular, $T(\mathcal{F}^k(t_k, \underline{s})) = |\operatorname{Re}(g^k(z))| > 0$, and hence one can see using a recursive argument that for all $0 \leq j \leq k$,

$$T(\mathcal{F}^j(t_k, \underline{s})) = \log\left(T(\mathcal{F}^{j+1}(t_k, \underline{s})) + 2\pi\{s_{j+1}\} + 1\right) > 0, \tag{5.1.14}$$

and so $\mathcal{F}^j(t_k, \underline{s})$ is indeed well-defined for all $j \leq k$. By definition of the map $\mathcal{C}_A$, $\operatorname{Re}(\mathcal{C}_A(\mathcal{F}^k(t_k, \underline{s}))) = \pm\operatorname{Re}(g^k(z)) \pm A$, where "$\pm$" equals "$+$" or "$-$" depending on whether $\sigma^k(\underline{s})$ belongs to $\mathbb{Z} \times \{R\}$ or $\mathbb{Z} \times \{L\}$. Moreover, by Observation 5.17, both $g^k(z), \mathcal{C}_A(\mathcal{F}^k(t_k, \underline{s})) \in F_{s_k}$, and hence by Proposition 5.9,

$$|\mathcal{C}_A(\mathcal{F}^k(t_k, \underline{s})) - g^k(z)| < 3\pi + A. \tag{5.1.15}$$

Note that for all $j \leq k$, since the second coordinate of $\mathcal{F}^j(t_k, \underline{s})$ equals $\sigma^j(\underline{s})$, $\Phi_{k-j}(\mathcal{F}^j(t_k, \underline{s})) = (g_{s_j}^{-1} \circ \cdots \circ g_{s_{k-1}}^{-1} \circ \mathcal{C}_A \circ \mathcal{F}^k)(t_k, \underline{s})$ and $g^j(z) = (g_{s_j}^{-1} \circ \cdots \circ g_{s_{k-1}}^{-1} \circ g^k)(z)$. Hence, using Observation 5.5 together with (5.1.12) and (5.1.15), by the same contraction argument as when showing (5.1.10), for any $j \leq k$,

$$|\mathcal{C}_A(\mathcal{F}^j(t_k,\underline{s})) - g^j(z)| \leq |\mathcal{C}_A(\mathcal{F}^j(t_k,\underline{s})) - \Phi_{k-j}(\mathcal{F}^j(t_k,\underline{s}))| + |\Phi_{k-j}(\mathcal{F}^j(t_k,\underline{s})) - g^j(z)| < 2\mu + \frac{3\pi + A}{2^{k-j}} < 2(\mu + 3\pi + A) =: \eta.$$

We note that the constant η does not depend on k. In particular, by taking $j = 0$ we see that t_k is uniformly bounded from above by a constant independent of k, and thus $t_k \nrightarrow \infty$ as $k \to \infty$. This means that there exists at least one finite limit point for the sequence $\{t_k\}_{k\geq 0}$, say $t \geq 0$, that by (5.1.14) satisfies $(t,\underline{s}) \in J(\mathcal{F})$. Since by (5.1.11) for each $j \geq 0$ it holds $g^j(\Phi(t,\underline{s})) = \Phi(\mathcal{F}^j(t,\underline{s}))$,

$$|g^j(\Phi(t,\underline{s})) - g^j(z)| \leq |\Phi(\mathcal{F}^j(t,\underline{s})) - \mathcal{C}_A(\mathcal{F}^j(t,\underline{s}))| + |\mathcal{C}_A(\mathcal{F}^j(t,\underline{s})) - g^j(z)| < 2\mu + \eta,$$

and this upper bound does not depend on j. Since $g^j(\Phi(t,\underline{s}))$ and $g^j(z)$ belong to the same fundamental domain F_{s_j} for each $j \geq 0$, we can use once more the same contraction argument to conclude that the points $\Phi(t,\underline{s})$ and z are equal. $\triangle$

To prove injectivity of Φ, note that if $\{(t,\underline{s}),(t',\underline{s})\} \subset J(\mathcal{F})$ for some $t \neq t'$, the orbits of $(t,\underline{s})$ and $(t',\underline{s})$ under $\mathcal{F}$ will eventually be far apart by definition of $\mathcal{F}$. Then, by (5.1.11) and (5.1.12), so will be the g-orbits of $\Phi(t,\underline{s})$ and $\Phi(t',\underline{s})$, and injectivity follows.

Since $J(\mathcal{F}) \cup \{\tilde{\infty}\}$ is by Proposition 5.15 a sequential space, and so is $\widehat{\mathbb{C}}$, the notions of continuity and sequential continuity for functions between these spaces are equivalent. Thus, using (5.1.13) we can extend Φ to a continuous map $\tilde{\Phi} : J(\mathcal{F}) \cup \{\tilde{\infty}\} \to J(g) \cup \{\infty\}$ by defining $\tilde{\Phi}(\tilde{\infty}) := \infty$. Then, $\tilde{\Phi}^{-1}$ is continuous as it is the inverse of a continuous bijective map on a compact space, and consequently, by respectively removing ∞ and $\tilde{\infty}$ from the domain and codomain of $\tilde{\Phi}^{-1}$, it follows that Φ^{-1} is also continuous. $\square$

Observation 5.22 (The embedding of $J(\mathcal{F})$ in $\mathbb{C}$ is a Cantor bouquet)**.** It follows from the proof of the claim above that for any $(t,\underline{s}) \in J(\mathcal{F})$, $\Phi(t,\underline{s}) \in J_{\underline{s}}$, see Observation 5.12. In particular, Φ acts as an order-preserving map from the exponentially bounded elements of $(\mathbb{Z} \times \{L,R\})^{\mathbb{N}}$ to $\operatorname{Addr}(g)$. Moreover, if g is any disjoint type cosine map, $J(g)$ is a Cantor bouquet, see Observation 5.2. Hence, $J(\mathcal{F})$ can be embedded on the plane using the homeomorphism Φ, and since $\Phi(J(\mathcal{F})) = J(g)$, $\Phi(J(\mathcal{F}))$ is a Cantor bouquet.

We can moreover relate the dynamics of any cosine map to the model using Theorem 5.21 together with [Rem09, Theorem 1.1]. In particular, the following result can be regarded as an analogue for cosine maps of the aforementioned result [Rem06, Theorem 4.2] for exponential maps.

Corollary 5.23 (Cosine model relates to all cosine maps)**.** *Let f be a cosine map. Then there exists a constant $R > 0$ and a quasiconformal map $\theta\colon \mathbb{C} \to \mathbb{C}$ such that*

$$\theta \circ \mathcal{F} = f \circ \theta \quad \textit{on} \quad J_R(\mathcal{F}) := \{z \in \mathbb{C}\colon T(\mathcal{F}^n(z)) > R \textit{ for all } n \geq 1\}.$$

Next, we shall combine Theorems 4.64 and 5.21 to construct a model for the dynamics of strongly postcritically separated maps in the cosine family. Firstly we note that for cosine maps, some conditions in the definition of strongly postcritically separated maps (Definition 3.3) are trivially satisfied, and thus they can be characterized the following way:

Proposition 5.24 (Cosine maps that are strongly postcritically separated)**.** *Let f be in the cosine family. Then the following are equivalent:*

(A) f is strongly postcritically separated;

(B) $P(f) \cap F(f)$ is compact and there exist constants $M > 0, K > 1$ and $\epsilon > 0$ such that for all $r > 0$ and all $z, w \in P_J$,

$$\#(P_J \cap A(r, Kr)) \leq M \quad \textit{and} \quad |z - w| \geq \epsilon \max\{|z|, |w|\}. \tag{5.1.16}$$

(C) Each critical value of f either converges to an attracting cycle, repelling periodic cycle, or converges to infinity sufficiently fast*, that is,* (5.1.16) *holds for f.*

Proof. By definition, (A) $\Rightarrow$ (B). For any cosine map f, $\mathrm{AV}(f) = \emptyset$ and any critical point has local degree equal to 2, see 5.1. In particular, f has bounded criticality in its Julia set. Moreover, since each cosine map has two critical values, for all $z \in J(f)$, $\#(\mathrm{Orb}^+(z) \cap \mathrm{Crit}(f)) \leq 2$. Hence, if f is in the cosine family and (B) holds for f, then all conditions in the definition of strongly postcritically separated maps (Definition 3.3) are satisfied, and so (B) $\Rightarrow$ (A). If (5.1.16) holds for f, then $P(f) \cap J(f)$ is discrete. In addition, since $f \in \mathcal{B}$, when P_J is discrete, P_F being compact is equivalent to all periodic cycles in $J(f)$ being repelling and $F(f)$ being a collection of attracting basins (see the proof of Lemma 3.5), and thus, (B) $\Leftrightarrow$ (C). ☐

Definition 5.25 (Model for cosine strongly postcritically separated functions)**.** Let g be any disjoint type cosine map and let $(J(g)_\pm, \tau_J)$ be a model space in the sense of Definition 4.55. Let $J(\mathcal{F})_\pm := J(\mathcal{F}) \times \{-,+\}$ and $\hat{\Phi} : J(\mathcal{F})_\pm \to J(g)_\pm$ given by $\hat{\Phi}(t, \underline{s}, *) := (\Phi(t, \underline{s}), *)$, where $\Phi\colon J(\mathcal{F}) \to J(g)$ is the homeomorphism from Theorem 5.21. Let us induce in $J(\mathcal{F})_\pm$ the topology

$$\tau_{\mathcal{F}} := \{\hat{\Phi}^{-1}(U) : U \in \tau_J\}.$$

We call $(J(\mathcal{F})_\pm, \tau_J)$ the *model space* for strongly postcritically separated cosine maps. Define $I(\mathcal{F})_\pm := I(\mathcal{F}) \times \{-,+\} \subset J(\mathcal{F})_\pm$ as a subspace equipped with the induced topology. The *model function* is $\tilde{\mathcal{F}} : J(\mathcal{F})_\pm \to J(\mathcal{F})_\pm$ given by $\widetilde{\mathcal{F}}(t, \underline{s}, *) := (\mathcal{F}(t, \underline{s}), *)$.

Observation 5.26 (Uniqueness of the model and continuity of $\widetilde{\mathcal{F}}$)**.** By Observation 4.56, any two model functions as defined above are conjugate. Hence, by Observation 5.2, the definition of the topology in $J(\mathcal{F})_\pm$ is independent of the cosine map g chosen. Alternatively, it is possible to induce the same topology in $(J(\mathcal{F})_\pm, \tau_J)$ without using the model $J(g)_\pm$ in a similar way we defined the topology of $J(g)_\pm$ in §4.5. Moreover, the model function $\widetilde{\mathcal{F}}$ is continuous, since by Theorem 5.21, it holds $\hat{\Phi} \circ \widetilde{\mathcal{F}} = \tilde{g} \circ \hat{\Phi}$, and so $\widetilde{\mathcal{F}}$ can be expressed as a composition of continuous functions.

Theorem 5.27 (Semiconjugacy between $\widetilde{\mathcal{F}}$ and strongly postcritically separated cosine maps)**.** *Let f be a strongly postcritically separated cosine map. Then, there exists a continuous surjective function*

$$\hat{\varphi}\colon J(\mathcal{F})_\pm \to J(f) \qquad \textit{so that} \qquad f \circ \hat{\varphi} = \hat{\varphi} \circ \widetilde{\mathcal{F}}$$

and $\hat{\varphi}(I(\mathcal{F})_\pm) = I(f)$. In addition, for every $z \in I(f)$, $\#\hat{\varphi}^{-1}(z) \leq 4$, and moreover, for each exponentially bounded $\underline{s} \in (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$ and $* \in \{-,+\}$, the restriction $\hat{\varphi}\colon \{(t, \underline{s}, *) : t \geq t_{\underline{s}}\} \to \overline{\Gamma}(\underline{s}, *)$ is a bijection.

Remark. We have implicitly stated in Theorem 5.27 that $\hat{\varphi}$ establishes a one-to-one correspondence between $\operatorname{Addr}(f)_\pm$ and the exponentially bounded elements in $(\mathbb{Z}\times\{L, R\})^{\mathbb{N}}\times\{-,+\}$, since we have stated that for each such $\underline{s} \in (\mathbb{Z}\times\{L, R\})^{\mathbb{N}}$, $\{(t, \underline{s}, *) : t \geq t_{\underline{s}}\}$ is mapped bijectively to $\overline{\Gamma}(\underline{s}, *) \subset J(f)$ for $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$. In particular, we are claiming that $\hat{\varphi}$ is an order-preserving continuous map. Compare to Observation 4.65.

Proof of Theorem 5.27. By Proposition 4.35, for some $\lambda \in \mathbb{C}$, $g := \lambda f$ is of disjoint type. By Observation 5.26, we can assume that $J(\mathcal{F})_\pm$ has been defined using the model $J(g)_\pm$. Let $\hat{\Phi} : J(\mathcal{F})_\pm \to J(g)_\pm$ be the homeomorphism from Definition 5.25. By Theorem 4.64, there exists a continuous surjective function

$\varphi : J(g)_\pm \to J(f)$ such that $f \circ \varphi = \varphi \circ \tilde{g}$ and so that $\varphi(I(g)_\pm) = I(f)$. Let $\hat{\varphi} := \varphi \circ \hat{\Phi} : J(\mathcal{F})_\pm \to J(f)$. Then

$$f \circ \hat{\varphi} = f \circ \varphi \circ \hat{\Phi} = \varphi \circ \tilde{g} \circ \hat{\Phi} = \varphi \circ \hat{\Phi} \circ \widetilde{\mathcal{F}} = \hat{\varphi} \circ \widetilde{\mathcal{F}},$$

as shown in the diagram:

$$\begin{tikzcd}
J(\mathcal{F})_\pm \arrow[r, "\widetilde{\mathcal{F}}"] \arrow[d, "\hat{\Phi}"] \arrow[dd, bend right=60, "\hat{\varphi}"'] & J(\mathcal{F})_\pm \arrow[d, "\hat{\Phi}"] \arrow[dd, bend left=60, "\hat{\varphi}"] \\
J(g)_\pm \arrow[r, "\tilde{g}"] \arrow[d, "\varphi"] & J(g)_\pm \arrow[d, "\varphi"] \\
J(f) \arrow[r, "f"] & J(f).
\end{tikzcd}$$

In particular, since by Theorem 5.21 $\Phi(I(\mathcal{F})) = I(g)$, by definition of the map $\hat{\Phi}$, it must occur that $\hat{\Phi}(I(\mathcal{F})_\pm) = I(g)_\pm$. Thus, $\hat{\varphi}(I(\mathcal{F})_\pm) = I(f)$. By Theorem 4.64, Observation 4.12 and since cosine maps have two critical values, $\#\varphi^{-1}(z) = \prod_{j=0}^{\infty} \deg(f, f^j(z)) \leq 4$ for all $z \in I(f)$. Since $\hat{\Phi}$ is a homeomorphism, the same bound applies to $\#\hat{\varphi}^{-1}(z)$. By Observation 5.22, $\Phi(\{(t, \underline{s}) : t \geq t_{\underline{s}}\}) = J_{\underline{s}}$, and hence $(\varphi \circ \hat{\Phi})(\{(t, \underline{s}, *) : t \geq t_{\underline{s}}\}) = \varphi(J_{(\underline{s},*)})$, where $J_{(\underline{s},*)} \subset J(g)_\pm$ is the set from 4.60. Then, by Theorem 4.64 and Observation 4.65, the last statement in the theorem follows. □

Remark. It is possible to construct directly the semiconjugacy from $J(\mathcal{F})_\pm$ to $J(f)$ without using Theorems 5.21 and 4.64. To do so, we would define $J(\mathcal{F})_\pm$ similarly as we defined the associated model $J(g)_\pm$ in §4.5 and the proof would be similar to the proof of Theorem 4.64, where instead of using that a disjoint type cosine map expands the Euclidean metric (Observation 5.5), one would use that f is strongly postcritically separated and thus, expands an orbifold metric (Theorem 1.1 and Corollary 3.13).

Identifications: extension to a conjugacy

Let f be a strongly postcritically separated cosine map and let $\hat{\varphi}\colon J(\mathcal{F})_\pm \to J(f)$ be the map from Theorem 5.27. If we define the equivalence relation in $J(\mathcal{F})_\pm$

$$a \sim b \iff \hat{\varphi}(a) = \hat{\varphi}(b), \tag{5.1.17}$$

then, since $\hat{\varphi}$ is continuous, by the Universal Property of Quotient Maps (see for example [Mun00, Theorem 2.22]), there exists a unique continuous function

$\tilde{\varphi}\colon J(\mathcal{F})_\pm/\sim \to J(f)$ such that the diagram

$$\begin{tikzcd} J(\mathcal{F})_\pm \arrow[r, "\hat{\varphi}"] \arrow[d, "\pi"] & J(f) \\ J(\mathcal{F})_\pm/\sim \arrow[ur, dashed, "\exists!\,\tilde{\varphi}"'] & \end{tikzcd}$$

commutes, where π is the projection function that takes each element to its equivalence class. In particular, since both $\hat{\varphi}$ and π are surjective, by definition $\tilde{\varphi}$ is bijective. By the commutative relation $f \circ \hat{\varphi} = \hat{\varphi} \circ \widetilde{\mathcal{F}}$ from Theorem 5.27, for any $a, b \in J(\mathcal{F})_\pm$,

$$\pi(a) = \pi(b) \Rightarrow \hat{\varphi}(\widetilde{\mathcal{F}}(a)) = f(\hat{\varphi}(a)) = f(\hat{\varphi}(b)) = \hat{\varphi}(\widetilde{\mathcal{F}}(b)) \Rightarrow \pi(\widetilde{\mathcal{F}}(a)) = \pi(\widetilde{\mathcal{F}}(b)),$$

and so, the function $h\colon J(\mathcal{F})_\pm/\sim \to J(\mathcal{F})_\pm/\sim$ given by $h(\pi(x)) := \pi(\widetilde{\mathcal{F}}(x))$ is well-defined. In particular, $\tilde{\varphi}$ conjugates h and f as shown in the following diagram:

$$\begin{tikzcd} J(\mathcal{F})_\pm \arrow[r, "\widetilde{\mathcal{F}}"] \arrow[d, "\pi"] \arrow[dd, bend right=60, "\hat{\varphi}"'] & J(\mathcal{F})_\pm \arrow[d, "\pi"] \arrow[dd, bend left=60, "\hat{\varphi}"] \\ J(\mathcal{F})_\pm/\sim \arrow[r, "h"] \arrow[d, "\tilde{\varphi}"] & J(\mathcal{F})_\pm/\sim \arrow[d, "\tilde{\varphi}"] \\ J(f) \arrow[r, "f"] & J(f). \end{tikzcd}$$

Observation 5.28 (Equivalence classes given by overlappings of $\overline{\Gamma}$-curves)**.** Since by Theorem 5.27, for each $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm$, or equivalently each exponentially bounded $\underline{s} \in (\mathbb{Z} \times \{L, R\})^{\mathbb{N}}$, $\hat{\varphi}(\{(t, \underline{s}, *) : t \geq t_{\underline{s}}\}) = \overline{\Gamma}(\underline{s}, *)$, in order to specify which elements belong to each equivalence class of $J(\mathcal{F})_\pm/\sim$, it suffices to determine the overlappings between the curves $\{\overline{\Gamma}(\underline{s}, *)\}_{(\underline{s},*)\in \operatorname{Addr}(f)_\pm}$. Recall from Theorem 4.64 that each of them is a canonical ray together with its endpoint. In particular, Proposition 4.10 provides information on the overlappings of canonical rays. However, in general, we do not have any information on whether they share their landing points, that is, on when canonical rays land together.

Our goal in this subsection is to make the relation "$\sim$" in (5.1.17) *explicit* for the map $f = \cosh$. That is, following Observation 5.28, we shall provide a combinatorial description of the equivalence classes of $J(\mathcal{F})_\pm/\sim$ in terms of the signed addresses of their images under $\tilde{\varphi}$. For a function $f \in \mathcal{B}$, the partition of a neighbourhood of infinity into fundamental domains, §2.4, is commonly regarded as a *static partition* in the sense that the curve δ and domain $D \Supset S(f)$ on its definition do not have dynamical meaning for f. In particular, dynamic rays of

f might cross the boundaries of fundamental domains infinitely often. Instead, given the further information that we have for our specific example, we can define a *dynamical partition*, so that the boundaries of the components are ray tails:

5.29 (Dynamical partition for $f = \cosh$)**.** Recall from Example 4.1 and 5.3 that for the function cosh, $J(f) = \mathbb{C}$ and $\mathrm{S}(f) = \mathrm{CV}(f) = \{-1, 1\}$. Moreover, the curves $\gamma_1 := \mathbb{R} \setminus (-\infty, 1)$ and $\gamma_{-1} := \mathbb{R} \setminus (-1, \infty)$ are ray tails joining 1 and -1 to ∞ whose forward orbits lie in $\mathbb{R}^+$. Let

$$X := \gamma_1 \cup \gamma_{-1}.$$

Since $\mathbb{C} \setminus X$ is simply connected and $S(f) \subset X$, by the Monodromy Theorem, each connected component of $f^{-1}(\mathbb{C} \setminus X)$ is a simply-connected domain, and the restriction of f to it is a conformal isomorphism into its image. More specifically, noting that the critical values of f are totally ramified, see 5.1, each connected component of $f^{-1}(\mathbb{C} \setminus X)$ is a horizontal strip

$$U_K := \{z \in \mathbb{C} \text{ such that } \pi K < \operatorname{Im} z < (K+1)\pi)\}$$

for some $K \in \mathbb{Z}$. We denote $\mathcal{U} := \{U_K\}_{K \in \mathbb{Z}}$. See Figure 15.

5.30 (Fixing signed addresses for $f = \cosh$)**.** Let us fix any bounded domain $D \supset [-1, 1] \supset S(f)$. Then, $f^{-1}(\mathbb{C} \setminus D)$ consists of two unbounded domains that do not contain the imaginary axis, since $f(i\mathbb{R}) = [-1, 1]$. Thus, we can choose $\delta := i\mathbb{R}^+$ and define fundamental domains for f as connected components of $f^{-1}(\mathbb{C} \setminus (\overline{D} \cup \delta))$. In particular, $f^{-1}(\delta)$ equals the collection of horizontal half-lines

$$\{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z = (1/2 + 2n)\pi i\},$$
$$\{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ and } \operatorname{Im} z = (-1/2 + 2n)\pi i\}$$

for all $n \in \mathbb{Z}$. Thus, each fundamental domain is contained in one of the half-strips

$$\begin{aligned} S_{(n,L)} &:= \{z : \operatorname{Re} z < 0, \operatorname{Im} z \in ((n - 1/2)\pi, (n + 3/2)\pi)\} \quad \text{or} \\ S_{(n,R)} &:= \{z : \operatorname{Re} z > 0, \operatorname{Im} z \in ((n - 3/2)\pi, (n + 1/2)\pi)\}. \end{aligned} \tag{5.1.18}$$

Compare to Example 4.1. For each $(n, *) \in (\mathbb{Z} \times \{-, +\})$, we denote by n_* the unique fundamental domain contained in $S_{(n,*)}$. Using these fundamental domains, we define the set of admissible external addresses $\mathrm{Addr}(f)$, and since $f \in \mathcal{CB}$ and hence is criniferous (Theorem 1.7), we can define a corresponding set of signed external addresses $\mathrm{Addr}(f)_\pm$ for f. See Definition 4.3. In particular, for the curves γ_1 and γ_{-1} from 5.29, recall that $f(\gamma_{-1}) \subset \gamma_1$, $f(\gamma_1) \subset \gamma_1$, and these curves belong to the fundamental domains $\gamma_1 \subset 0_R$ and $\gamma_{-1} \subset 0_L$. In particular, each of these curves equals two canonical tails with opposite sign, as they do not contain preimages of critical points, see Proposition 4.10. Hence, $\gamma_1 \subset \varGamma(\overline{0_R}, *)$ and $\gamma_{-1} \subset \varGamma(0_L\overline{0_R}, *)$ for both $* \in \{-,+\}$. See Figure 14.

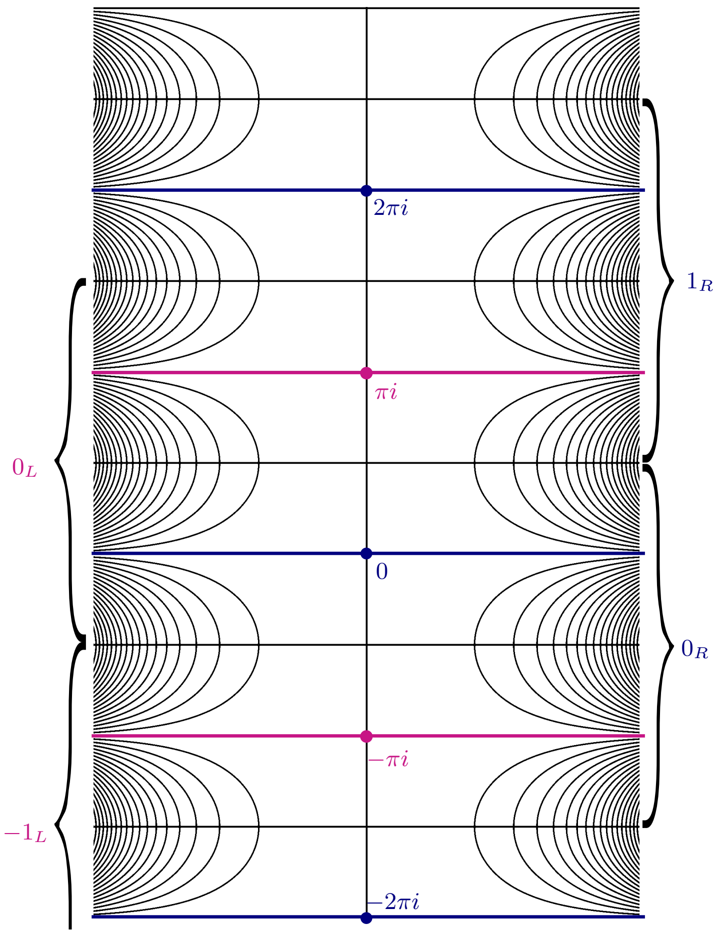

Figure 14: Partition of the plane into fundamental domains and itinerary components for cosh. In particular, each strip of height π between two coloured lines is an itinerary domain. Some fundamental domains are indicated by keys. Also, displayed are the first (coloured lines and imaginary axis), second (other horizontal lines) and third (rest of curves) successive preimages of the real line.

[†]Picture originally generated by Rempe-Gillen and modified for this thesis.

The dynamical partition from 5.29 will aid us on determining if any of the dynamic rays of f land together. This is because since the boundaries of the elements in $\mathcal{U}$ are canonical tails, as we shall see, no other canonical tails can *cross* them. More precisely, in the next proposition we assign to each curve $\{\overline{\varGamma}(\underline{s}, *)\}_{(\underline{s},*)\in \text{Addr}(f)_\pm}$ a unique element of $\mathcal{U}$:

Proposition 5.31 (Each canonical ray is in the closure of a unique $U \in \mathcal{U}$)**.** *For each* $(\underline{s}, *) \in \text{Addr}(f)_\pm$*, there exists a unique component* $U \in \mathcal{U}$ *such that*

$\overline{\Gamma}(\underline{s}, *) \subset \overline{U}$. *We denote*

$$U(\underline{s}, *) := U.$$

Proof. Since as described in 5.30, the set X from 5.29 is formed by four canonical tails that overlap pairwise, and in addition f is totally ramified, exactly four canonical tails meet at each critical point of $f^{-1}(X)$, and their union is a connected component of this set. More precisely, following the analysis in 5.30, $f^{-1}(\gamma_1)$ is the collection of all the horizontal lines $\{2\pi i K\mathbb{R}\}_{K\in\mathbb{Z}}$, and so they contain the critical points $2\pi K i$ for all $K \in \mathbb{Z}$. Analogously, $f^{-1}(\gamma_{-1})$ is the collection of the horizontal lines $\{2(K+1)\pi i\mathbb{R}\}_{K\in\mathbb{Z}}$ with critical points at $2(K+1)\pi i$ for each $K \in \mathbb{Z}$. Hence, it follows that for each $K \in \mathbb{Z}$ and $* \in \{-,+\}$,

$$\begin{aligned} &2\pi K\mathbb{R}^- \subset \Gamma(K_L\overline{0_R}, *) \qquad \text{and } 2\pi(K+1)\mathbb{R}^- \subset \Gamma(K_L 0_L\overline{0_R}, *), \\ &2\pi K\mathbb{R}^+ \subset \Gamma(K_R\overline{0_R}, *) \qquad \text{and } 2\pi(K+1)\mathbb{R}^+ \subset \Gamma((K+1)_R 0_L\overline{0_R}, *). \end{aligned} \tag{5.1.19}$$

We claim that each of the canonical rays displayed in (5.1.19) belongs to the closure of exactly one component of $\mathcal{U}$: to see this, let us for example consider the curves $\Gamma(K_R\overline{0_R}, *)$ for some $K \in \mathbb{Z}$ and both $* \in \{-,+\}$. Then, since $\Gamma(K_R\overline{0_R}, -)$ is a nested sequence of left-extended canonical tails and by Proposition 4.10 it can only intersect the boundaries of the elements of $\mathcal{U}$ in the subcurve $2K\pi i\mathbb{R}^+$, we conclude that $\Gamma(K_R\overline{0_R}, -) \subset \overline{U_{K-1}}$. Similarly, $\Gamma(K_R\overline{0_R}, +) \subset \overline{U_K}$, and arguing analogously for the rest of curves in (5.1.19), the claim follows.

Let us denote by $\operatorname{Addr}(\partial\mathcal{U})$ the set of all signed addresses of the curves in (5.1.19) for all $K \in \mathbb{Z}$ and $* \in \{-,+\}$. Then, any $(\underline{s}, *) \in \operatorname{Addr}(f)_\pm \setminus \operatorname{Addr}(\partial\mathcal{U})$ belongs to some interval $\mathcal{I}$ of the form

$$((K_L 0_L\overline{0_R}, -), (K_L\overline{0_R}, +)), \quad ((K_R\overline{0_R}, -), ((K+1)_R 0_L\overline{0_R}, +)) \tag{5.1.20}$$

for some $K \in \mathbb{Z}$. Hence, since $\gamma^0_{(\underline{s},*)} = \gamma^0_{\underline{s}}$ and the Julia constituents of the endpoints of each interval in (5.1.20) belong to two different connected components of the boundary of some element of $\mathcal{U}$, by (2.4.9), $\gamma^0_{(\underline{s},*)} \Subset U_K$ for some $K \in \mathbb{Z}$. By Observation 4.65, the map $\varphi : J(g)_\pm \to J(f)$ from Theorem 4.64 is a continuous map that preserves the orders of $\operatorname{Addr}(g)_\pm$ and $\operatorname{Addr}(f)_\pm$, and thus $\varphi(J_{(\underline{\tau},\star)}) \subset U_K$ for all $(\underline{\tau}, \star) \in \mathcal{I}$. In particular, $\overline{\Gamma}(\underline{s}, *) = \varphi(J_{(\underline{s},*)}) \Subset U_K$. □

Remark. Instead of proving Proposition 5.31 using continuity of the map φ from Theorem 4.64, one could had instead used the characterization of canonical rays as nested sequences of left or right extensions, in a similar way as we did in §4.2 to prove Proposition 4.22 and Lemma 4.23.

Definition 5.32 (Itineraries for $f = \cosh$)**.** For each $(\underline{s}, *) \in \operatorname{Addr}(f)_{\pm}$, we define the *itinerary* of $(\underline{s}, *)$ as the infinite sequence

$$\operatorname{itin}(\underline{s}, *) := U(\underline{s}, *)U(\sigma(\underline{s}), *)U(\sigma^2(\underline{s}), *)\ldots.$$

Moreover, for each $N \in \mathbb{N}$ we denote by $\operatorname{itin}_N(\underline{s}, *)$ the restriction of $\operatorname{itin}(\underline{s}, *)$ to its first N terms.

Observation 5.33 (Itineraries of points)**.** Since by Proposition 5.31, for each $(\underline{s}, *) \in \operatorname{Addr}(f)_{\pm}$, $f(\overline{\Gamma}(\underline{s}, *)) \subset \overline{\Gamma}(\sigma(\underline{s}), *)$, if $z \in \overline{\Gamma}(\underline{s}, *)$ and $\operatorname{itin}(\underline{s}, *) = U_0U_1\ldots$, then $f^i(z) \subset U_i$ for all $i \geq 0$.

Proposition 5.34 (Dynamic rays of cosh do not land together)**.** *There are no two dynamic rays of* cosh *landing together.*

Proof. By Observation 4.14, it suffices to prove that there are no two canonical rays landing together. With that aim, let $\Gamma(\underline{s}, *)$ and $\Gamma(\underline{\tau}, \star)$ be two different canonical rays, that is, $(\underline{s}, *) \neq (\underline{\tau}, \star)$, and let $p_{(\underline{s},*)}$ and $p_{(\underline{\tau},\star)}$ be their respective landing points. If $\Gamma(\underline{s}, *)$ and $\Gamma(\underline{\tau}, \star)$ land together, i.e. $p_{(\underline{s},*)} = p_{(\underline{\tau},\star)}$, then by Proposition 5.31 and Observation 5.33, $\operatorname{itin}(\underline{s}, *) = \operatorname{itin}(\underline{\tau}, *) = U_0U_1\ldots$. Moreover, for each $i \geq 0$, $f^i(p_{(\underline{s},*)}) = f^i(p_{(\underline{\tau},\star)})$ must belong to the interior of U_i, since by 5.30 (see also Example 4.1), the boundaries of the elements of $\mathcal{U}$ are formed by canonical tails that are contained in dynamic rays. For the same reason, $i\mathbb{R}^+$ and $f^{-1}(i\mathbb{R}^+)$ do not contain any landing points, as they are formed by pieces of ray tails. See 5.35 for more details. Then, for each $i \geq 0$, $f^i(p_{(\underline{s},*)}) = f^i(p_{(\underline{\tau},\star)})$ belongs to a half-strip of the form

$$\begin{aligned} HS^i_{(k,L)} &:= \left\{ z : \operatorname{Re} z \leq 0, \operatorname{Im} z \in \left(\frac{(i+4k)\pi}{4}, \frac{(i+1+4k)\pi}{4} \right) \right\} \quad \text{or} \\ HS^i_{(k,R)} &:= \left\{ z : \operatorname{Re} z \geq 0, \operatorname{Im} z \in \left(\frac{(i+4k)\pi}{4}, \frac{(i+1+4k)\pi}{4} \right) \right\} \end{aligned} \tag{5.1.21}$$

for some $k \in \mathbb{Z}$ and $0 \leq i \leq 3$. However, each of the strips in (5.1.21) intersects a single fundamental domain, see (5.1.18) and Figure 14, which contradicts $(\underline{s}, *) \neq (\underline{\tau}, \star)$. $\square$

Finally, we provide a combinatorial description of the overlapping of canonical rays in terms of their signed addresses, which by Observation 5.28 and Proposition 5.34 suffices to describe the equivalence classes in $J(\mathcal{F}) \setminus \sim$.

5.35 (Overlapping of canonical tails for cosh)**.** Recall that by Proposition 4.10, for all $(\underline{s}, *) \in \operatorname{Addr}(f)_{\pm}$ such that $\Gamma(\underline{s}, *) \cap \operatorname{Orb}^-(\operatorname{Crit}(f)) = \emptyset$,

$$\Gamma(\underline{s}, -) = \Gamma(\underline{s}, +).$$

Hence, all other overlappings occur between the preimages of the canonical tails that contain $\operatorname{Crit}(f)$. Recall from 5.29 and 5.30 that $\operatorname{Crit}(f) = \{\pi i K : K \in \mathbb{Z}\}$, and each critical point belongs exactly to four canonical rays containing $\operatorname{Crit}(f)$. Namely, we saw in (5.1.19) that

$$\begin{aligned}
\mathit{\Gamma}(K_L\overline{0_R}, -) &= \mathit{\Gamma}(K_L\overline{0_R}, +) && \text{in } 2K\pi i\mathbb{R}^- && \text{for all } K \in \mathbb{Z}.\\
\mathit{\Gamma}(K_R\overline{0_R}, -) &= \mathit{\Gamma}(K_R\overline{0_R}, +) && \text{in } 2K\pi i\mathbb{R}^+ && \text{for all } K \in \mathbb{Z}.\\
\mathit{\Gamma}(K_L 0_L\overline{0_R}, -) &= \mathit{\Gamma}(K_L 0_L\overline{0_R}, +) && \text{in } 2(K+1)\pi i\mathbb{R}^+ && \text{for all } K \in \mathbb{Z}.\\
\mathit{\Gamma}((K+1)_R 0_L\overline{0_R}, -) &= \mathit{\Gamma}((K+1)_R 0_L\overline{0_R}, +) && \text{in } 2(K+1)\pi i\mathbb{R}^- && \text{for all } K \in \mathbb{Z}.
\end{aligned}$$

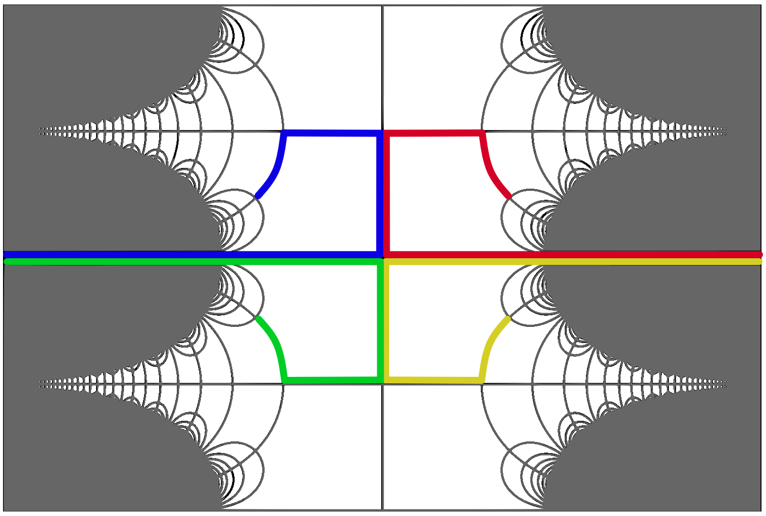

Figure 15: Picture showing four canonical tails of cosh that contain the critical point 0. More precisely, these are parts of the canonical rays $\mathit{\Gamma}(0_L\overline{0_R}, -)$ (in blue), $\mathit{\Gamma}(\overline{0_R}, +)$ (in red), $\mathit{\Gamma}(0_L\overline{0_R}, +)$ (in green) and $\mathit{\Gamma}(\overline{0_R}, -)$ (in yellow).

†Picture originally generated by Rempe-Gillen and modified for this thesis.

By the extending criterion, since each canonical ray is a nested sequence of left or right extended curves, further identifications between the canonical rays above, occur at the connected components of $f^{-1}([0,1])$ and $f^{-1}([-1,0])$. More precisely, if for each $\pm \in \{-,+\}$ we denote

$$V_K(\pm) := \{z \in \mathbb{C} : \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z \in [K\pi i, (K \pm 1/2)\pi i]\},$$

then we have the following identifications:

$$\begin{aligned}
\mathit{\Gamma}(K_L\overline{0_R}, \mp) &= \mathit{\Gamma}(K_R\overline{0_R}, \pm) && \text{in } V_{2K}(\pm) && \text{for all } K \in \mathbb{Z}.\\
\mathit{\Gamma}(K_L 0_L\overline{0_R}, \mp) &= \mathit{\Gamma}((K+1)_R 0_L\overline{0_R}, \pm) && \text{in } V_{2K+1}(\pm) && \text{for all } K \in \mathbb{Z},
\end{aligned}$$

where $\mp = +$ when $\pm = -$ and vice-versa, $\mp = -$ when $\pm = +$. See Figure

15. Next, by Proposition 4.10, any further overlappings between canonical rays occur at preimages of the overlappings already stated. More specifically, since we have already described all overlappings occurring at the boundaries of the itinerary components, all remaining ones must occur between canonical tails with the same N-th itinerary (that is, each of the first N-th points in their forward orbit lie in the same components of $\mathcal{U}$) that are mapped under f^N to the coordinate axes. Given the geometry of the fundamental domains, contained in half-strips of height π, see (5.1.18) and Figure 14, providing the identifications happening at the preimages of the positive imaginary axis, allows us to express identifications solely using external addresses, since any further identifications must occur at the intersection of a fundamental domain with a component of $\mathcal{U}$, and hence, the corresponding first entries on the signed addresses of rays overlapping are the same. More specifically, let us denote

$$I^K_{P_R}(\pm) := f^{-1}(V_K(\pm)) \cap P_R \quad \text{and} \quad I^K_{P_L}(\pm) := f^{-1}(V_K(\pm)) \cap P_L.$$

Then, for all $K \in \mathbb{Z}^+$ and $P \in \mathbb{Z}$,

$$\begin{aligned}
\Gamma((P+1)_R K_L \overline{0_R}, \mp) &= \Gamma(P_R K_R \overline{0_R}, \pm) && \text{in } I^{2K}_{P_R}(\pm)\\
\Gamma((P+1)_R K_L 0_L \overline{0_R}, \mp) &= \Gamma(P_R (K+1)_R 0_L \overline{0_R}, \pm) && \text{in } I^{2K+1}_{P_R}(\pm)\\
\Gamma(P_L K_L \overline{0_R}, \mp) &= \Gamma((P+1)_L K_R \overline{0_R}, \pm) && \text{in } I^{K}_{P_L}(\pm)\\
\Gamma(P_L K_L 0_L \overline{0_R}, \mp) &= \Gamma((P+1)_L (K+1)_R 0_L \overline{0_R}, \pm) && \text{in } I^{2K+1}_{P_L}(\pm).
\end{aligned}$$

See Figure 15. Moreover, for all other canonical rays, if I is some bounded curve in $J(f)$, then

$$\Gamma(\underline{s}, *) = \Gamma(\underline{\tau}, \star) \text{ in } I \iff \begin{cases} \exists\, n > 0 : s_j = \tau_j \text{ for all } j \leq n \quad \text{and} \\ \Gamma(\sigma^n(\underline{s}), *) = \Gamma(\sigma^n(\underline{\tau}), \star) \text{ in } f^n(I). \end{cases} \tag{5.1.22}$$

Example 5.36 (Overlappings for the map $\cosh^2$)**.** Seeking an example where a critical value is mapped to another critical value, we consider the function

$$f(z) := \cosh^2(z) := \cosh(z) \cdot \cosh(z) = \frac{e^{2z} + e^{-2z}}{4} + \frac{1}{2}.$$

Even if strictly speaking this function is not in the cosine family, it is in their parameter space, since $\cosh^2$ is conjugate to $\frac{e}{2}e^w + \frac{e^{-1}}{2}e^{-w}$ via $\psi(z) := 2z - 1$. The function $\cosh^2$ has already appeared in the literature, namely in [RS12], where it is shown that $I(f)$ (and in fact its *fast escaping set*) is connected. The

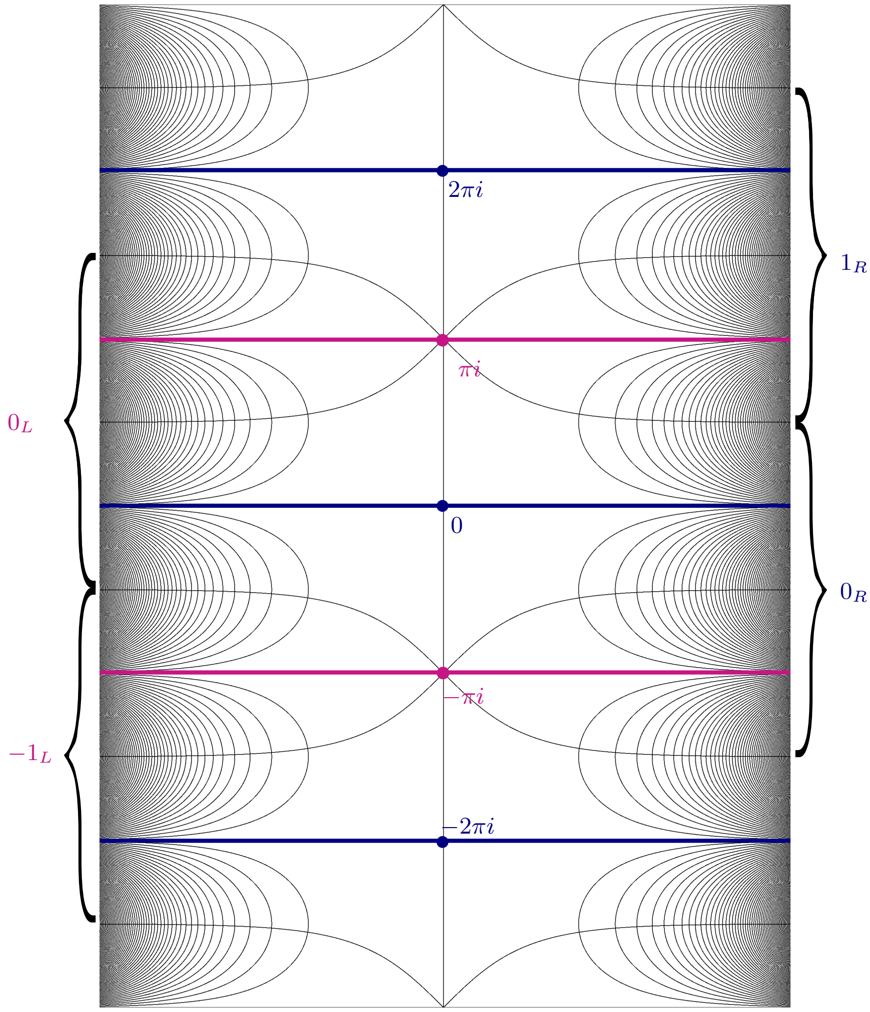

Figure 16: Partition of the plane into fundamental domains and itinerary components for $\cosh^2$. In particular, each strip of height π between two coloured lines is an itinerary domain. Some fundamental domains are indicated by keys. Also, displayed are the first (coloured lines and imaginary axis), second (other curves that meet at $\{K\pi i : K \in \mathbb{Z}\}$) and third (rest of curves) iterated preimages of the real line.

†Picture originally generated by Rempe-Gillen and modified for this thesis.

function f is πi-periodic, and $S(f) = \mathrm{CV}(f) = \{0, 1\}$, with $f(0) = 1 \in I(f)$ and $\mathrm{Orb}^+(1) \subset \mathbb{R}$. The critical points of f are $f^{-1}(1) = \{K\pi i : K \in \mathbb{Z}\}$ and $f^{-1}(0) = \{(K+1/2)\pi i : K \in \mathbb{Z}\}$. Similar as for the map cosh, we can join the critical values to infinity using the ray tails $\gamma_1 := \mathbb{R} \setminus (-\infty, 1)$ and $\gamma_0 := \mathbb{R} \setminus (0, \infty)$, and define a dynamical partition for f as the connected components of $\mathbb{C} \setminus f^{-1}(\gamma_0 \cup \gamma_1)$, which are horizontal half-strips of $\pi/2$-height and we call *itinerary domains*. See Figure 16. By choosing a bounded domain D containing $[0, 1]$ and $\delta := i\mathbb{R}^+ \setminus D$, we can define fundamental domains for f as the connected components of $f^{-1}(\mathbb{C} \setminus (\overline{D} \cup \delta))$. In particular, we label as 0_R the component that contains an unbounded subset of $\mathbb{R}^+$, and as 0_L the component that contains an unbounded subcurve of $\mathbb{R}^-$. Additionally, we label as K_R and K_L their respective $K\pi i$-translates. See Figure

16. With this nomenclature, the following identifications between canonical rays occur:

$$\Gamma(K_R\overline{0_R},-)=\Gamma(K_R\overline{0_R},+) \qquad \text{in } K\pi i\mathbb{R}^+ \qquad \text{for all } K\in\mathbb{Z}.$$
$$\Gamma(K_L\overline{0_R},-)=\Gamma(K_L\overline{0_R},+) \qquad \text{in } K\pi i\mathbb{R}^- \qquad \text{for all } K\in\mathbb{Z}.$$

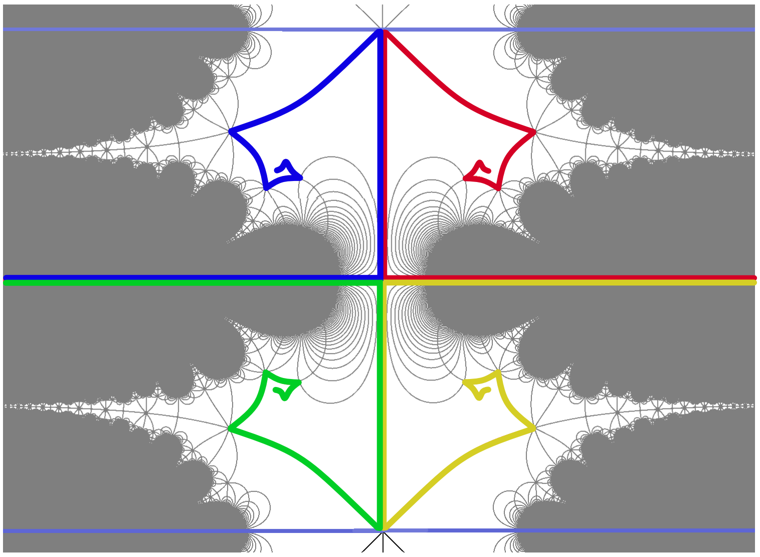

Figure 17: Picture showing four canonical tails of $\cosh^2$ that contain the critical point 0. More precisely, these are parts of the canonical rays $\Gamma(0_L\overline{0_R},-)$ (in blue), $\Gamma(\overline{0_R},+)$ (in red), $\Gamma(0_L\overline{0_R},+)$ (in green) and $\Gamma(\overline{0_R},-)$ (in yellow).

†Picture originally generated by Rempe-Gillen and modified for this thesis.

The main difference between the overlappings between the canonical rays of cosh and the overlappings between those of $\cosh^2$, is that since the critical points in $f^{-1}(0)$ are mapped to a critical point, each of them belongs to eight ray tails rather than four, and moreover, both singular values belong to the canonical rays $\Gamma(K_R\overline{0_R},*)$ for both $*\in\{-,+\}$. Compare Figures 15 and 17. Then, we further have the identifications

$$\Gamma(K_L\overline{0_R},\mp)=\Gamma(K_R\overline{0_R},\pm) \quad \text{in } [K\pi i,(1\pm 1/2)K\pi i]=:V_K(\pm) \quad \text{for all } K\in\mathbb{Z}.$$

If for each $\pm\in\{-,+\}$ and $K\in\mathbb{Z}$ we let

$$I^K_{P_R}(\pm):=f^{-1}(V_K(\pm))\cap P_R \quad\text{and}\quad I^K_{P_L}(\pm):=f^{-1}(V_K(\pm))\cap P_L,$$

then for all $K\in\mathbb{Z}^+$ and $P\in\mathbb{Z}$,

$$\Gamma((P+1)_RK_L\overline{0_R},\mp)=\Gamma(P_RK_R\overline{0_R},\pm) \qquad \text{in } I^K_{P_R}(\pm)$$
$$\Gamma((P+1)_LK_R\overline{0_R},\pm)=\Gamma(P_LK_L\overline{0_R},\mp) \qquad \text{in } I^K_{P_L}(\pm).$$

Any further identifications between canonical rays occur within the intersection of a fundamental domain and an itinerary domain, and hence can be expressed using (5.1.22). Moreover, arguing as for cosh, no two dynamic rays of $\cosh^2$ land together, see the proof of Proposition 5.34.

5.2 Questions and remarks

In this section we briefly discuss several questions that naturally arise from the results in this thesis.

Expansion for postcritically separated maps

The conditions imposed in the definition of (strongly) postcritically separated maps are aimed to prove Theorem 1.1, that is, to achieve orbifold expansion for a class of functions with unbounded postsingular set. In particular, our goals to reach for a transcendental entire function f were the following:

(1) To construct a pair of *orbifolds dynamically associated* to f, that is, so that their underlying surfaces contain $J(f)$ and $f\colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map.

(2) To show that f is *expanding* with respect to the orbifold metric of $\mathcal{O}$. That is, $\|Df(z)\|_{\mathcal{O}} \geq \Lambda$ for some $\Lambda > 1$ whenever $z, f(z) \in \mathcal{O}$.

In order to achieve (1), we need to assume bounded criticality in $J(f)$: as pointed out in [MB12, Proposition 3.6], f cannot map as a branched cover in any preimage of a neighbourhood of an asymptotic value $a \in J(f)$, and hence f cannot be an orbifold covering map for any neighbourhood of a. Moreover, if the local degree of critical points was not uniformly bounded, then there could exist a point $p \in J(f)$ and an infinite sequence $\{p_n\}_{n\geq 0} \subset f^{-1}(p)$ such that $\deg(f, p_{i+1}) > \deg(f, p_i)$ for all $i \geq 0$, and that would prevent f from being an orbifold covering map, since it would require the ramification value of p to be a multiple of $\deg(f, p_i)$ for all i, and hence not a natural number.

Seeking to obtain both (1) and (2), we have demanded $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ to be holomorphic. A requirement for this is that if S is the underlying surface of $\mathcal{O}$, then $J(f) \subset f^{-1}(S) \subseteq S$. By the condition imposed on the Fatou set of postcritically separated maps, that is, $P(f) \cap F(f)$ being compact, all postsingular points in $F(f)$ lie in a collection of attracting basins (Lemma 3.5), and so we can define S satisfying the requirement. As discussed at the beginning of §3.3, $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ being holomorphic demands all points in P_J to be ramified points in $\mathcal{O}$. Hence, we need to assume that P_J is discrete. In addition, for $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ to be holomorphic, it is required the existence of a constant c such that $\#(\mathrm{Orb}^+(z) \cap \mathrm{Crit}(f)) \leq c$ for all $z \in J(f)$, a condition imposed in the definition of strongly postcritically separated maps. See (3.3.1).

The function f belonging to class $\mathcal{B}$ guarantees that once the associated orbifolds are defined, the boundary of $\widetilde{\mathcal{O}}$ and $\hat{\mathcal{O}}$, denoted $\mathbb{B}^{\hat{\mathcal{O}}}_{\widetilde{\mathcal{O}}}$, has "enough points spread on the plane". See Proposition 3.11. However, we note that in our definition of associated orbifolds (Definition and Proposition 3.10), all points in $f^{-1}(P(f) \setminus S(f))$ are ramified points in $\widetilde{\mathcal{O}}$, and hence this assumption a priori might not be essential. Item (c) in Definition 3.3 of strongly postcritically separated maps is specifically targeted to get (2) by using Theorem 1.13. Thus, since our results on orbifold metrics might be opened to improvement, so do these assumptions on f.

Consequently, with sharper estimates on orbifold metrics or by modifying the definition of associated orbifolds, we conjecture that some conditions in our definitions might be dropped without compromising orbifold expansion:

Conjecture 5.37 (Orbifold expansion for postcritically separated maps)**.** Theorem 1.1 generalizes to postcritically separated functions with bounded criticality on their Julia set. That is, for every postcritically separated f with bounded criticality in $J(f)$, there exist a constant $\Lambda > 1$ and a pair of hyperbolic orbifolds $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ such that $f \colon \widetilde{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map, $\|Df(z)\|_{\mathcal{O}} \geq \Lambda > 1$ and $J(f)$ is contained in the underlying surfaces of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.

More generally, one could ask for *expansion* in neighbourhoods of Julia sets with respect to some metric that might not necessarily be an orbifold one. For example, if in the definition of postcritically separated maps we relax the hypothesis of $P(f) \cap F(f)$ being compact to $S(f) \cap F(f)$ being compact, then $F(f)$ might contain parabolic basins, as occurs for parabolic transcendental maps [Alh19], or more generally for geometrically finite maps [MB10, Proposition 2.5]. A transcendental entire map f is *parabolic* if $S(f) \subset F(f)$ and $P(f) \cap J(f)$ is a collection of finitely many parabolic points. Recall that f is *geometrically finite* if $S(f) \cap F(f)$ is compact and $P(f) \cap J(f)$ is finite. If in addition f has bounded criticality on its Julia set, then f is called *strongly geometrically finite*. In particular, strongly geometrically finite maps belong to the class $\mathcal{B}$ and generalize strongly subhyperbolic ones.

When f has parabolic orbits, the function cannot be uniformly expanding near a parabolic point for any conformal metric (see [Alh19, Introduction and §4.3]), nor it is possible to define dynamically associated orbifolds like in (1), since there is no neighbourhood $S \supset J(f)$ such that $f^{-1}(S) \subset S$. This obstacle is overcome for parabolic maps [Alh19] (resp. strongly geometrically finite maps [ARGS19]) by defining a hyperbolic (resp. orbifold) metric in a domain S such that $J(f) \subset \overline{S}$ and so that all parabolic points belong to ∂S. Then, this metric is modified in neighbourhoods of parabolic points by assuming its density is constant on them, and using the local information of how f acts near parabolic points.

Observation 5.38 (Expansion in the presence of parabolic points)**.** It seems plausible that combining the techniques in [ARGS19, Alh19] with the orbifold results in this thesis, *expansion* can be achieved for a more general class. Namely, for strongly postcritically maps for which the assumption on the Fatou set is relaxed to $S(f) \cap F(f)$ being compact, and thus containing parabolic points. In particular, most results in this thesis are likely to hold for this more general class.

Fatou components and quasidiscs

Recall that the key step in the proof of Theorem 1.2 was, provided Theorem 1.2(b) holds, to obtain in Theorem 3.17 different characterizations of boundedness of *periodic* Fatou components. In an analogous version of Theorem 3.17 for hyperbolic functions [BFRG15, Theorem 1.10], it is shown that in that case, all periodic Fatou components are not just Jordan domains but quasidiscs. In particular, this implies that every bounded Fatou component of a hyperbolic entire function is a quasidisc [BFRG15, Corollary 1.11]. As in our proof of Theorem 3.17, the authors show in the proof of [BFRG15, Theorem 1.10] that whenever Theorem 3.17(7) holds for a hyperbolic function f and a periodic Fatou component D, ∂D is a Jordan curve. Then, they conclude that D is in fact a quasidisc by using Douady-Hubbard straightening theorem [DH85, Theorem 1, p. 296] in a suitable neighbourhood $\Omega \supset D$ such that $\Omega \cap P(f) \setminus D = \emptyset$.

If more generally f is a strongly postcritically separated map, the existence of a neighbourhood Ω with that property it is no longer guaranteed, since ∂D might contain points in $P(f) \cap J(f)$. Hence, their argument does not apply to our functions. Nonetheless, we suspect that Theorem 3.17 can be strengthened to show that under any of the conditions stated, all periodic Fatou components are quasidiscs, and in particular that [BFRG15, Corollary 1.11] generalizes to strongly postcritically separated functions in class $\mathcal{B}$:

Conjecture 5.39. Let $f \in \mathcal{B}$ strongly postcritically separated. Then every bounded Fatou component of f is a quasidisc.

We believe this conjecture might be proven to hold using the definition of quasicircle combined with Theorem 1.13. Recall that a *quasicircle* is the image of the unit circle $\mathbb{S}^1$ under a quasisymmetric map. For Jordan curves in $\mathbb{C}$, quasicircles are characterized as those curves that satisfy a bounded turning condition [Ahl63]. That is, a Jordan curve $\gamma \subset \mathbb{C}$ is *bounded turning* (or C-bounded turning) if there exists a constant $C > 1$ such that for each pair of points $z, w \in \gamma$,

$$\operatorname{diam}(\gamma[z,w]) < C|z-w|,$$

where $\gamma[z,w] \subset \gamma$ denotes the arc of smaller (Euclidean) diameter between z and w. Then, for a periodic Fatou component that is a Jordan domain, one might be able to cover its boundary with balls of some fix radius, and by taking pullbacks and using Corollary 3.13, show that its boundary is bounded turning.

Explosion of endpoints

Once it is known for a transcendental entire function f that its Julia set is a collection of landing dynamic rays, an interesting question to ask is what is the topology of the set $E(f)$ of the endpoints of dynamic rays. Here, by *endpoint* we mean the landing point of a dynamic ray. The seminal work in this direction dates from 1990 and is due to Mayer [May90]. For exponential maps $f_a \colon \mathbb{C} \to \mathbb{C}$ with $f_a(z) := e^z + a$ and all parameters $a \in (-\infty, 1)$, $J(f_a)$ is a Cantor bouquet [DK84], and hence a union of hairs and endpoints. Mayer proved that for any such exponential map, its set of endpoints $E(f_a)$ is *totally separated*, while $E(f_a) \cup \{\infty\}$ is a connected set. In other words, infinity is an *explosion* point for $E(f_a)$.

This surprising result was generalized in [ARG17] for further parameters, and thus more exponential maps. Moreover, the authors of [ARG17] show for each of those maps f_a that infinity is an explosion point for the smaller set of all *escaping endpoints*, i.e. the set $\tilde{E}(f_a) := E(f_a) \cap I(f)$. In addition, in that same paper, some of their results are extended outside the exponential family:

Theorem 5.40 ([ARG17, Theorem 1.9])**.** *Let $f \in \mathcal{B}$ be a finite composition of functions of finite order, or more generally satisfying any of the conditions in Proposition 4.39. Then $\tilde{E}(f) \cup \{\infty\}$ is connected. If, additionally, f is hyperbolic, then $\tilde{E}(f)$ is totally separated.*

The proof of the first part of Theorem 5.40 relies on the existence of *absorbing Cantor bouquets* in the Julia sets of the maps considered, [BJR12, Theorem 1.6]. It follows from the results in this thesis that [BJR12, Theorem 1.6] generalizes to all functions in $\mathcal{CB}$:

Proposition 5.41 (Absorbing Cantor bouquets)**.** *Let $f \in \mathcal{CB}$. Then, for each $R > 0$, there exists a Cantor bouquet $X \subset J(f)$ such that $f(X) \subset X$ and*

$$J_Q(f) \subset X \subset J_R(f)$$

for all $Q > R$ sufficiently large.

Proof. Let $g := \lambda f$ such that g is of disjoint type. Since $f \in \mathcal{CB}$, $J(g)$ is a Cantor bouquet. For each $R > 0$, if $\pi_R \colon J(g) \to J_R(g)$ is the function defined in (4.3.3), then by Observation 4.30, for all $M > R$ sufficiently large $Y := \pi_R(J(g)) \supset J_M(g)$. Moreover, by Theorem 4.31, π_R is a continuous function, and hence, since $J(g)$ is a Cantor bouquet, so is Y. In addition, by Observation 4.32, it holds that $g(Y) \subset Y$. By Corollary 4.45 and the proof of Proposition 4.49, for all R large enough, there exists a continuous map $\theta \colon J_R(g) \to J(f)$ that is a homeomorphism onto its image and so that $J_Q(f) \subset \theta(Y) \subset J_{\lambda^2 R}(f)$ for some $M > R$ and $f(\theta(Y)) \subset \theta(Y)$. In particular, $X := \theta(Y)$ is a Cantor bouquet and the proposition follows. $\square$

The second statement of Theorem 5.40 only holds for maps that in addition are hyperbolic because its proof uses that for each f hyperbolic, there exists a semiconjugacy between f and a disjoint type map λf, that in particular has a Julia Cantor bouquet, on their Julia sets. Additionally, the semiconjugacy is a homeomorphism between their escaping sets [Rem09, Theorem 1.4]. Thus, we believe that we could use Theorem 1.8 to generalize Theorem 5.40 the following way:

Conjecture 5.42. If $f \in \mathcal{CB}$, then $\tilde{E}(f) \cup \{\infty\}$ is connected. If in addition f is strongly postcritically separated, then $\tilde{E}(f)$ is totally separated.

Recently, further research has been carried out in the topology of endpoints. The main objects of study in [ERG18], and subsequently in [ES19], are different subsets of endpoints. In particular, [ERG18, Theorem 1.2] states that for the exponential maps that Mayer studied, i.e., f_a with $a \in (-\infty, -1)$, contrary to the result for escaping endpoints, $J(f_a) \setminus I(f_a) \cup \{\infty\}$ is totally separated. In [ES19], a more general class of functions is considered, and for any such function f, the set of escaping *meandering endpoints* is defined as $\tilde{E}_M(f) := \tilde{E}(f) \setminus A(f)$, where $A(f)$ denotes the *fast escaping set*. For the definition of fast escaping set, see for example the article [BH99], where it was first studied. Then, [ES19, Theorem 1.4] states that if f is hyperbolic and of finite order, then $\tilde{E}_M(f) \cup \{\infty\}$ is totally separated. The idea of their proof relies again in the semiconjugacy provided by [Rem09, Theorem 1.4] between f and a disjoint type with Julia Cantor bouquet, that they show to be a homeomorphism between fast escaping sets [ES19, Proposition 7.2]. We again think that Theorem 5.40 could be used to generalize this result, but since the fast escaping set has not been studied in this thesis, we rather ask:

Question 5.43. Let $f \in \mathcal{CB}$ strongly postcritically separated. Is $\tilde{E}_M(f) \cup \{\infty\}$ totally separated?

Landing of dynamic rays and unbounded postsingular sets

As commented above, for our results on orbifold expansion, the assumption of bounded criticality in the Julia set is essential to define associated orbifolds. When this assumption is dropped, as occurs in the exponential family when the asymptotic value is in the Julia set, a very different phenomenon can occur. In fact, for certain parameters in the exponential family, the corresponding maps have dynamic rays whose accumulation sets are indecomposable continua [DJMR05, DJ02]. Moreover, it was shown in [Rem07] that if the singular value of an exponential map is on a dynamic ray or it is the landing point of a ray, then there exist uncountably many dynamic rays whose accumulation set is an indecomposable continuum. In that paper, Rempe-Gillen notes that "*the presence of a dynamic ray landing at an asymptotic value was the driving factor in our proof. In particular, one would expect the following dichotomy: if f is a postcritically finite entire function of finite order, then*

⋆ if f has an asymptotic value, some dynamic ray of f accumulates on an entire dynamic ray, and conversely,

⋆ if f has only critical values, then every dynamic ray of f lands and every point in $J(f)$ is either on a ray or the landing point of such ray".

However, later, Bergweiler, Fagella and Rempe-Gillen constructed a hyperbolic function with only two critical values and no asymptotic values for which the Julia set is not locally connected [BFRG15, Example 1.7]. This was achieved by creating Fatou components of large diameter, that in turn exist due to the presence of critical points of extremely high local degree that approximate the behaviour of an asymptotic value. This suggests the following question:

Question 5.44. Does there exist an entire function with only critical values and containing indecomposable continua as accumulation sets of dynamic rays? If so, what conditions on the criticality of the function should be imposed? Which role does the order of the function play on such existence, i.e., can a very slow order of growth "compensate" the high criticality so that dynamic rays still land?

In addition, studying other transcendental entire functions with unbounded postsingular set, with maybe some of their postsingular points in the bungee set, seems a natural continuation of this work as well as an interesting path to explore.

Beyond dynamic rays: dreadlocks

In this thesis we have studied transcendental entire functions that contain dynamic rays on their escaping sets. However, this does not always occur, as for example, a function with bounded postsingular set is constructed in [RRRS11] so that its escaping set, and indeed its Julia set, contains no arcs. In other words, not all functions in class $\mathcal{B}$ are criniferous. Still, recall that in §2.4 we defined external addresses, and hence Julia constituents, for all functions in class $\mathcal{B}$. That is, for these functions, we are able to assign combinatorics to points whose orbit is sufficiently large, and group them in sets with convenient dynamical properties, as for example they contain connected unbounded subsets of points that escape uniformly to infinity. See Theorem 2.23.

A fair strategy to equip the Julia and escaping sets of a function in class $\mathcal{B}$ with symbolic dynamics is to *extend* Julia constituents so that the union of these extensions covers these sets. In this direction, the concept of *dreadlocks* is developed in [BRG17] for functions in class $\mathcal{B}$ with bounded postsingular set. In particular, since these functions do not have escaping singular values, *fundamental hands* can be defined just as preimages of fundamental domains, and are called *fundamental tails* [BRG17, Definition 3.4]. Then, in that paper, the authors Rempe-Gillen and Benini *extend* each Julia constituent to, in a certain sense, a maximal set using fundamental tails, and then call the intersection of this maximal set with the escaping set a *dreadlock*, [BRG17, Definition 4.2]. In particular, dreadlocks are connected, unbounded, and whenever the function is criniferous, they are dynamic rays [BRG17, Proposition 4.10 and Corollary 5.4].

In the presence of escaping singular values, an analogous definition of dreadlocks is not obvious to us. Note that in this thesis we have also *extended* the subsets "$J^\infty_{\underline{s}}$" of Julia constituents, but only for criniferous functions (without asymptotic values). This is because then, by Theorem 2.30, the sets "$J^\infty_{\underline{s}}$" are *curves*, and hence, if any of their preimages contains a critical point p, the topological structure of that preimage in a neighbourhood of p is simple: it consists of a collection of (topological) radial segments centred at p. See Figure 7. Hence, this allows us to establish a criterion to systematically *extend* them, so that the resulting sets are still dynamic rays, see Definition 4.5. However, it follows from [RG16] that in general, Julia constituents can be topologically much more complicated, as for example, their closure may be a hereditarily indecomposable continuum. Thus, a priori, it is unclear to us how to define extensions around critical points in a more general (topological) setting.

Question 5.45. Is it possible to define *dreadlocks* for functions in class $\mathcal{B}$ with unbounded postsingular set? In particular, for those that do not contain asymptotic values in their Julia set?

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